INTRODUCTORY PHYSICS

Lötz Strauss

INTRODUCTORY PHYSICS

Lötz Strauss
D Sc, THED, L Akad, Sci Nat

Professor, Physics Department
University of Pretoria

Published by the author

The original text in this book was prepared by the EmTeX version of LaTeX.
The sketches were made using the CorelDraw computer program.
The camera-ready pages were prepared by the author on a laser printer.

The digital copy was prepared by Cayden de Wit using the original LaTeX from the first edition. The LaTeX was modified such that it could compile on pdfLaTeX. Changes were applied from a physical copy of the second edition provided by Heinrich Klopper. The preparation of the manuscript (specifically, the figures) was aided by Dr. Walter Meyer.

ISBN 0-86979-944-4
First edition (1994)
Second (revised) edition (1995)

Publisher: Prof L Strauss, Department of Physics, University of Pretoria
PRETORIA 0002

Reproduced and bound by the University Printer, on the campus of the University of Pretoria.

Contents

Preface

Introductory Physics is designed to help first year college and university students to make a smooth transition from school physics to that which is studied at tertiary level. A prerequisite for the use of this book is a thorough knowledge and skill in the fields of elementary differential and integral calculus, elementary vector algebra with the inclusion of scalar and vector products and also elementary vector analysis (the differentiation and integration of simple vector functions).

The material is presented in such a way that the necessary *concepts and principles* (as opposed to topics) may be mastered in the correct order. In this presentation the traditional (and unnecessary) division of physics into watertight compartments such as mechanics, optics, electricity, etc. is avoided. When the concept of force is treated, all forces are treated and the same applies to the concept of energy. This approach leads to a better understanding of nature and it avoids much duplication which is the usual result of topic-based courses.

Much emphasis is placed on the solution of problems and to help the student, the text contains numerous problems which are solved in full detail. In the solution of problems students are required to know the definitions of basic concepts and be able to apply the required mathematical techniques skilfully. Many years of teaching experience has shown that this approach is economical as regards the field which can be treated in a given interval of time and it also nurtures a student attitude which enables them to work and think independently when they have to solve problems which they have not encountered previously.

My heartfelt thanks to my dear wife for proofreading at level one (correcting her husband's appallingly poor English spelling and grammar). The most difficult task was the detailed checking of the manuscript which is a translation of the original version in Afrikaans. This immense task was performed by my dear and respected friend Etienne Malherbe. For the elimination of errors missed by the other readers, the final draft was revised by Danie Steyn, another dear friend who was deeply involved in the preparation of the original Afrikaans version.

Some of his suggestions led to major changes in the original text.

Many typographical errors which occurred in the first edition have been eliminated in the second. This was made possible by the comments which were supplied by Walter Meyer (tutor) and Bernhard Bhmer (student). Their contributions are appreciated.

My thanks to my Maker for the privilege, opportunity and ability to plan and accomplish this task. *Soli Deo Gloria!*

Department of Physics *Lötz Strauss*
University of Pretoria
January 1995

Chapter 1

THE KINEMATICS OF A POINT

1.1 Units and dimensions

In this text SI units (*Le Système International d'Unités*) are used. This system is fully described in the publication M 33 of the South African Buro of Standards (July 1977).

The system is based on seven base units which are treated on page 4 of the above-mentioned publication. Here only three, those of time, length and mass are mentioned.

Physical concept	**Unit**	**Symbol**
time	second	s
length	metre	m
mass	kilogram	kg

It is a **decimal** system (i.e. based on powers of ten) and the following prefixes are used to indicate multiple and fractional parts of units and which represent powers of ten:[1]

[1]The American system differs in some respects

Factor	**Factor in words**	**Prefix**	**Symbol**
10^{18}	trillion	exa-	E
10^{15}	billiard	peta-	P
10^{12}	billion	tera-	T
10^{9}	milliard	giga-	G
10^{6}	million	mega-	M
10^{3}	thousand	kilo-	k
10^{-3}	thousandth	milli-	m
10^{-6}	millionth	micro-	μ
10^{-9}	milliardth	nano-	n
10^{-12}	billionth	pico-	p
10^{-15}	billiardth	femto-	f
10^{-18}	trillionth	atto-	a

The extent to which the basic physical quantities, mass (M), length (L), and time (T) are represented in the physical quantity, is known as the **dimensions** of that quantity. In writing, the dimensions of a quantity are expressed in the following notation:

$$
\begin{aligned}
[\text{area}] &= [\mathrm{L}^2] \\
[\text{speed}] &= [\mathrm{L\,T}^{-1}] \\
[\text{density}] &= [\mathrm{M\,L}^{-3}] \\
[\text{force}] &= [\mathrm{M\,L\,T}^{-2}] \\
[\text{pressure}] &= [\mathrm{M\,L}^{-1}\ \mathrm{T}^{-2}] \\
[\text{work}] &= [\mathrm{M\,L}^{2}\,\mathrm{T}^{-2}]
\end{aligned}
$$

1.2 Position, displacement and track length

To specify the position of a point in three-dimensional space, three numbers are required. These numbers are known as **co-ordinates** and for a given position in space they will depend on the **frame of reference** or **co-ordinate system** relative to which the position is specified. Although a large variety of different frames of reference exists, there are three which occur more frequently than the others in the description and solution of problems in physics. They are (i) Cartesian co-ordinates (also known as rectangular co-ordinates), (ii) spherical polar co-ordinates and (iii) cylindrical polar co-ordinates.

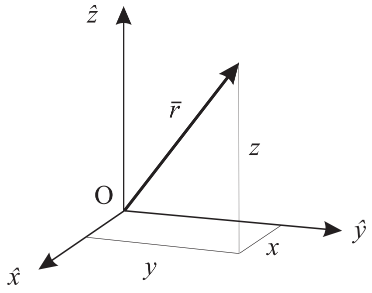


Fig.1.2-1

When using Cartesian co-ordinates, the position of a point is specified by means of three numbers x, y and z as shown in Figure 1.2-1. Some users prefer the notation x_1, x_2 and x_3, which also has merit for some applications. These three numbers specify the positions along three mutually perpendicular axes and they are measured from the common zero position which is called the **origin** of the system. Usually these axes are chosen to form a right-handed system. The three co-ordinates form a vector which is called **position vector** and which will very often be written in the following manner:

$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z} \qquad 1.2(1)$$

In this notation $\hat{x}, \hat{y}$ and $\hat{z}$ are known as the base vectors of the Cartesian system. Matrix notation, (x, y, z), is sometimes preferable, and is in all respects equivalent to 1.2(1).

In **spherical polar co-ordinates**, the position of a point is specified by means of the three numbers r, θ and ϕ. Here r is the **distance from the origin** (always a positive number), θ the **polar angle** or **co-latitude** and ϕ the **latitude**, the **azimuth angle** or, in short, **azimuth**. These three quantities are shown in Figure 1.2-2(a). Base vectors are very seldom used in polar co-ordinates. The exception is the unit vector $\hat{r}$ which always points **radially** away from the origin. By using simple trigonometry, the reader can easily verify the following relationships between the Cartesian and polar co-ordinates for a given position.

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta \qquad 1.2(2)$$

The reader should take note that some people prefer to exchange the roles of θ

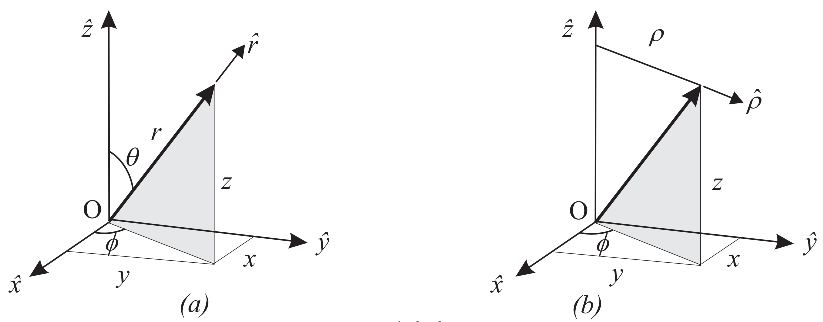


Fig.1.2-2

and ϕ. This requires an investigation before an unknown text is read. It should be clear that a misunderstanding of notation could lead to utter confusion.

Using **cylindrical polar co-ordinates**, the position of a point is specified by means of the three numbers ρ, ϕ and z. Here ρ (always a positive number) is the **distance from the z-axis**, ϕ the **latitude** or **azimuth angle** and z the **z-co-ordinate** as it was used with Cartesian co-ordinates. The only useful base vector in this case, is $\hat{\rho}$ which points radially away from the z-axis. The relationships between the cylindrical and Cartesian co-ordinates follow from Figure 1.2-2(b).

$$x = \rho \cos \phi, \qquad y = \rho \sin \phi, \qquad z = z \tag{1.2(3)}$$

In this text Cartesian co-coordinates will be used nearly exclusively in the development of the kinematics of a point. It is of prime importance to remember that the initial point of a position vector, $\bar{r}$, always coincides with the origin of the frame of reference which is used. This fact allows one to determine the **distance between a given point and the origin** of the frame of reference by calculating the magnitude of its position vector.

$$r = |\bar{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}} \tag{1.2(4)}$$

If *all three* the components of the position vector are constant the point is **at rest**. On the other hand, if at least one is a function of the time, t, the point is **in motion**. In general, x, y and z will be time dependent for a point which is in motion and its position vector may be written as

$$\bar{r} = \bar{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \tag{1.2(5)}$$

As time increases, each of x, y and z will change and thus also $\bar{r}$ of which the terminal point will describe a **space curve** which is the track of the point in motion. Each value of the time, t, corresponds with *one and only one* value of each of x, y and z, i.e. with only one position on the space curve. This is in accordance with the fact that no point can be at two different positions at the same time. The three functions $x = x(t)$, $y = y(t)$ and $z = z(t)$ are independent of each other but are connected by the **parameter** t. They are known as the **parametric equations** of the space curve and together they describe the motion of the point.

Examples:

1. $\bar{r} = \bar{r}(t) = (0)\hat{x} + (40t)\hat{y} + (-5t^2 + 20t)\hat{z}$ m in which the time, t, is measured in seconds.

The parametric equations of the space curve are as follows: $x = 0$, $y = 40t$ and $z = -5t^2 + 20t$. The space curve is a parabola through the origin in the plane $x = 0$. The Cartesian equation of this parabola follows by elimination of the parameter t from $y = 40t$ and $z = -5t^2 + 20t$:

$z = -(\frac{1}{320})y^2 + (\frac{1}{2})y$. This is a different but perfectly equivalent manner in which to describe the space curve.

2. $\bar{r} = \bar{r}(t) = (4\cos 2\pi t)\hat{x} + (4\sin 2\pi t)\hat{y} + (0)\hat{z}$ m in which the time, t, is measured in seconds.

The parametric equations of the space curve are $x = 4\cos 2\pi t$, $y = 4\sin 2\pi t$ and $z = 0$. This represents a circle with radius 4 m of which the centre is at the origin of the frame of reference. It lies in the plane $z = 0$ (the x-y-plane) and the position vector describes one revolution per second. The Cartesian equation of the circle may be calculated by eliminating t from the parametric equations. This is most easily accomplished by first calculating the squares of x and y.

$$x^2 = 16\cos^2 2\pi t, \quad y^2 = 16\sin^2 2\pi t$$

and then calculating their sum

$$\begin{aligned} x^2 + y^2 &= 16(\cos^2 2\pi t + \sin^2 2\pi t) \\ \text{or} \qquad x^2 + y^2 &= 16 \end{aligned}$$

3. $\bar{r} = \bar{r}(t) = (4\cos 8\pi t)\hat{x} + (4\sin 8\pi t)\hat{y} + (12t)\hat{z}$ m in which the time, t, is measured in seconds.

The parametric equations of the space curve are as follows: $x = 4\cos 8\pi t$, $y = 4\sin 8\pi t$ and $z = 12t$. These equations describe a **right circular helix** around the z-axis. The radius of the helix is 4 m. The period is 0,25 s and it has a **pitch** of 3 m per revolution. The initial position ($t = 0$) is (4, 0, 0) and since it requires 0,25 s to complete one revolution, it moves $0,25 \times 12$ m $= 3$ m in the $\hat{z}$-direction in that time.

A three-dimensional space curve can be described only by means of a set of three parametric equations. Examples (1) and (2) allowed the calculation of Cartesian equations because the space curves were on flat surfaces and the frames of reference chosen in such a way that each curve could be described by two non-trivial parametric equations.

If the position of a point changes, it is said to be **displaced**. The **displacement** over a given interval of time, Δt, is defined as the change of position during that interval. If $\bar{r}_1$ is the position at time t_1 and $\bar{r}_2$ that at a *later* time t_2, the displacement, $\Delta\bar{r}$, is given by:

$$\Delta\bar{r} = \bar{r}_2 - \bar{r}_1 \qquad 1.2(6)$$

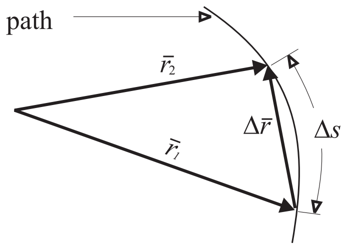


Fig.1.2-3

Figure 1.2-3 shows a graphical representation of a displacement vector. It should be clear from the sketch that the displacement vector supplies no information about either the path followed by the point or the distance travelled during the time interval Δt. The only information that can be derived from it is (i) the shortest distance between the beginning and the end-point of its journey and (ii) the direction of the end-point relative to the beginning.

Example: The initial position of a point is $\bar{r}_1 = 2\hat{x} - 3\hat{y} + 4\hat{z}$ m, and its final position $\bar{r}_2 = 4\hat{x} + 2\hat{y} + 2\hat{z}$ m. Calculate (a) its displacement, (b) the magnitude of the displacement.

(a) $\Delta\bar{r} = \bar{r}_2 - \bar{r}_1 = (4\hat{x} + 2\hat{y} + 2\hat{z}) - (2\hat{x} - 3\hat{y} + 4\hat{z}) = 2\hat{x} + 5\hat{y} - 2\hat{z}$ m

(b) $\mid \Delta r \mid = (4 + 25 + 4)^{\frac{1}{2}} = 5,74$ m

The **path length** for a moving point is the total distance travelled along the space curve on which it is moving. This is always a positive number which is usually larger than, but at the very least equal to the magnitude of the displacement vector. It is emphasized once again that in general, the displacement supplies no information about the path length. Only for the special case of a point moving along a straight line in one direction, will the path length be equal to the magnitude of the displacement vector.

An **infinitesimal displacement** is always tangential to the space curve (see Figure 1.23 and extrapolate to the case where $\Delta\bar{r}$ becomes very small) and the following will always be true:

$$ds = \mid d\bar{r} \mid = [(dx)^2 + (dy)^2 + (dz)^2]^{\frac{1}{2}} \qquad 1.2(7)$$

in which the symbol s, is used to indicate the distance travelled along the space curve from a chosen position on it. The distance s thus measures the path length. (This is what the odometer of a motor car measures.) It will therefore be correct to say that the vectors $\overline{ds}$ and $d\bar{r}$ are the same. For a finite displacement Δs is the distance measured along the space curve and will in general not be equal to $|\Delta\bar{r}|$ which is the magnitude of a vector pointing directly from the beginning to the end of the journey.

A finite path length may be calculated by means of a definite integral:

$$\Delta s = \int_P^Q ds = \int_P^Q \mid d\bar{r} \mid = \int_P^Q [(dx)^2 + (dy)^2 + (dz)^2]^{\frac{1}{2}} \qquad 1.2(8)$$

in which P and Q symbolically refer to the co-ordinates of the beginning and end-point of the path respectively. This integral is known as a **line integral** and can only be calculated if the parametric equations of the space curve are known.

In chapter 2, the work done by a force on a moving body will also be calculated by means of a line integral. Line integrals will be treated in that section and the calculation of path length will have to be left until this technique is mastered.

1.3 Average speed and average velocity

The **average speed** of a moving point *during a specified interval of time* is defined as the total distance travelled divided by the time interval.

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{\Delta s}{\Delta t} \qquad 1.3(1)$$

Average speed is a scalar and can only be a positive quantity.

The **average velocity** of a moving point *during a given interval of time* is the resultant displacement divided by the time interval.

$$\text{average velocity} = \frac{\text{resultant displacement}}{\text{time taken}} = \frac{\Delta \bar{r}}{\Delta t} \qquad 1.3(2)$$

Average velocity is a vector quantity. The SI units of both average speed and average velocity are $\mathrm{m\,s^{-1}}$ and their dimensions are $[\mathrm{L\,T^{-1}}]$.

In general average speed and average velocity are not very useful concepts since they are usually only valid for the time intervals for which they were calculated.

Problem: A person travels from A to B at a constant speed of 20 $\mathrm{km\,h^{-1}}$ and back along the same route at a constant speed of 40 $\mathrm{km\,h^{-1}}$. The time taken to reverse the direction may be disregarded. Calculate the average speed over the entire journey. (Answer: 26,67 $\mathrm{km\,h^{-1}}$)

1.4 Speed, velocity and acceleration

The **speed** of a moving point at a given instant is the rate at which the distance travelled increases at that time. The following mathematical form of the definition explains the concept well and is useful when making calculations:

$$\text{speed} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \qquad 1.4(1)$$

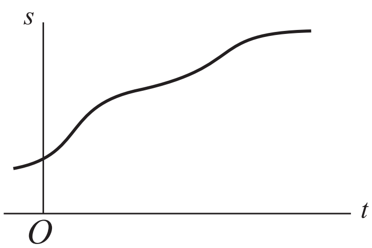


Fig.1.4-1

Geometrically, speed may be represented as the gradient of a distance-time graph, $s = s(t)$. The distance travelled by a moving point is also called the **path length** and is always a positive quantity which can never decrease. At the very least it can remain constant when the point is at rest. The result is that the gradient of this graph can only be positive or, at the very least, equal to zero. Fig 1.4-1 shows an example of an $s = s(t)$-graph.
The point $t = 0$ on the time-axis represents the instant when the measurement of the time was commenced. Negative values of t represent points in time before $t = 0$. Although the value of t can never decrease, negative values have a legitimate meaning.

The **velocity**, $\bar{v}$, of a moving point at a given point in time, is the rate at which its position changes at that instant. It may also be described as the rate at which it is displaced. Expressed in mathematical form, its meaning becomes clearer and a way of calculating it is also supplied:

$$\text{velocity} = \bar{v} = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t} = \frac{d\bar{r}}{dt} \qquad 1.4(2)$$

Because it consists of three independent components which are functions of time, it is generally not possible to represent a velocity vector graphically as a function of time. For this three separate graphs are required.

Figure 1.4-2 shows an example of an $x = x(t)$-graph. For motion in three dimensions, the graphs of $y = y(t)$ and $z = z(t)$ would also be required for a complete description of the time-dependent position vector, $\bar{r} = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$.

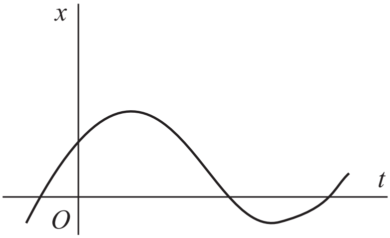


Fig.1.4-2

As is the case with $x = x(t)$, each component of $\bar{r}$ may be positive, negative or zero and may also increase, decrease or remain constant. We now know that velocity is a vector quantity which, in general, has three components. In order to establish a graphical interpretation of velocity, it is helpful to rewrite Equation 1.4(2) as

$$\bar{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

This vector equation is equivalent to the following three equations for the components of the velocity:

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}, \qquad v_x = \frac{dz}{dt}$$

From these, it should be clear that v_x is the gradient of the $x = x(t)$ graph, v_y that of the $y = y(t)$ graph and v_z that of the $z = z(t)$ graph.

From Equation 1.2(7) it is known that $ds = |d\bar{r}|$ and we may thus write that

$$\text{speed} = \frac{ds}{dt} = \frac{|\,d\bar{r}\,|}{dt} = \left|\frac{d\bar{r}}{dt}\right| = |\,\bar{v}\,| = v \qquad 1.4(3)$$

From this it can be seen that speed is the magnitude of the velocity vector. If the velocity is known in terms of its components, the speed may be calculated as follows:

$$v = [(v_x)^2 + (v_y)^2 + (v_z)^2]^{\frac{1}{2}} \qquad 1.4(4)$$

Since the sum of the squares of real numbers can only be zero if each and every one is zero, the speed of a point can only be equal to zero if each of the components of its velocity is zero.

The SI units of speed and velocity are $\mathrm{m\,s^{-1}}$ and the dimensions of both are $[\mathrm{L\,T^{-1}}]$.

If the velocity of a moving point changes, it can do so in the following ways: (i) the speed may change (increase or decrease), (ii) the direction may change, (iii) the speed and the direction may change. It is perfectly in order to speak of the *increase* or *decrease* of the speed of a moving point, but this does not apply to velocity. We may, however, refer to a *change* in velocity. (*This statement applies to all vector quantities.*)

If the velocity of a moving point changes, the point is said to **accelerate**. The **acceleration** at a given point in time is the rate at which the velocity changes. In mathematical form, it may be expressed as

$$\text{acceleration} = \bar{a} = \lim_{\Delta t \to 0} \frac{\Delta \bar{v}}{\Delta t} = \frac{d\bar{v}}{dt} \qquad 1.4(5)$$

This vector equation may also be written as

$$\bar{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} = \frac{dv_x}{dt}\hat{x} + \frac{dv_y}{dt}\hat{y} + \frac{dv_z}{dt}\hat{z}$$

which is equivalent to the following three equations:

$$a_x = \frac{dv_x}{dt}, \qquad a_y = \frac{dv_y}{dt}, \qquad a_z = \frac{dv_z}{dt}$$

In general, three graphs will be required to represent the acceleration as a function of time. The a_x will be the gradient of the $v_x = v_x(t)$ graph. Identical relationships hold for the other components.

The SI units of acceleration are $\mathrm{m\,s^{-2}}$ and its dimensions are $[\mathrm{L\,T^{-2}}]$.

The relationships between position, velocity and acceleration may be summarised as follows:

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right) = \frac{d^2\bar{r}}{dt^2} \qquad 1.4(6)$$

The following is a briefer notation for the above and is useful for some applications:

$$\bar{a} = \dot{\bar{v}} = \ddot{\bar{r}} \qquad 1.4(7)$$

The latter notation has the disadvantage that it does not indicate what the independent variable is. It should only be used when no possibility for confusion exists.

1.5 The solution of problems in kinematics

With the definitions of kinematic quantities and a knowledge of the necessary mathematics, it is now possible to solve problems in kinematics *in which time is the independent variable*. There are, in fact, only three different problems of this type and they are fully discussed in sections 1.5.1 to 1.5.3. The problems are treated by means of examples. A schematic summary is given in section 1.5.4 and the reader is advised to make a brief study of this summary before reading the following sections.

1.5.1 First kind: The position vector, $\bar{r} = \bar{r}(t)$, is known

Example: The position of a point relative to a Cartesian frame of reference is given by

$$\bar{r} = (3t^2 - 2t + 3)\hat{x} + (1,5t^2 - 2t + 2)\hat{y} + (-t^2 - t + 1)\hat{z} \text{ m}$$

in which the time, t, is measured in seconds.

(a) Calculate its initial position.

The word *initial* implies that $t = 0$. If this value is substituted for t in the position vector, the following is obtained:

$$\bar{r}(0) = 3\hat{x} + 2\hat{y} + \hat{z} \text{ m}$$

(b) Calculate the initial distance from the origin.
From Equation 1.2(4):

$$r(0) = |\,\bar{r}(0)\,| = (3^2 + 2^2 + 1)^{\frac{1}{2}} = (14)^{\frac{1}{2}} = 3,742 \text{ m}$$

(c) Calculate the velocity as a function of the time.
Equation 1.4(2) gives the definition of velocity. From this follows:

$$\begin{aligned} \bar{v} = \frac{d\bar{r}}{dt} &= \frac{d}{dt}\left[(3t^2 - 2t + 3)\hat{x} + (1,5t^2 - 2t + 2)\hat{y} + (-t^2 - t + 1)\hat{z}\right] \\ &= (6t - 2)\hat{x} + (3t - 2)\hat{y} + (-2t - 1)\hat{z} \text{ m s}^{-1} \end{aligned}$$

(d) Calculate the initial velocity.

To calculate the initial velocity, $t = 0$, is substituted in $\bar{v} = \bar{v}(t)$.

$$\bar{v}(0) = -2\hat{x} - 2\hat{y} - \hat{z} \text{ m s}^{-1}$$

(e) Calculate the initial speed.

Since speed is the magnitude of the velocity vector, the magnitude of the initial velocity has to be calculated.

$$v(0) = |\,\bar{v}(0)\,| = [(-2)^2 + (-2)^2 + (-1)^2]^{\frac{1}{2}} = 3 \text{ m s}^{-1}$$

(f) Calculate the speed as a function of the time.

Once again the fact is used that speed is the magnitude of the velocity vector.

$$v(t) = |\,\bar{v}(t)\,| = [(6t-2)^2 + (3t-2)^2 + (-2t-1)^2]^{\frac{1}{2}} = (49t^2 - 32t + 9)^{\frac{1}{2}} \ \mathrm{m\,s^{-1}}$$

By equating t to zero, the answer of (e) can be checked.

(g) Calculate the acceleration.

This may be calculated directly from the definition in Equation 1.4.(5).

$$\begin{aligned} \bar{a} = \frac{d\bar{v}}{dt} &= \frac{d}{dt}\,[(6t-2)\hat{x} + (3t-2)\hat{y} + (-2t-1)\hat{z}] \\ &= 6\hat{x} + 3\hat{y} - 2\hat{z} \ \mathrm{m\,s^{-2}} \end{aligned}$$

(h) Calculate the displacement between $t = 0$ s and $t = 1$ s.

To calculate this, the definition in Equation 1.2(6) may be applied directly. Note that the first position vector has to be subtracted from the second, always in this order.

$$\Delta\bar{r} = \bar{r}(1) - \bar{r}(0) = (4\hat{x} + 1,5\hat{y} - \hat{z}) - (3\hat{x} + 2\hat{y} + \hat{z}) = \hat{x} - 0,5\hat{y} - 2\hat{z} \ \mathrm{m}$$

1.5.2 Second kind: The velocity vector, $\bar{v} = \bar{v}(t)$, is known

By definition $\quad \bar{v} = \dfrac{d\bar{r}}{dt}$ and therefore we may write $\bar{r} = \displaystyle\int \bar{v}dt$ 1.5(1)

which is equivalent to the following three equations:

$$x = \int v_x dt, \qquad y = \int v_y dt, \qquad z = \int v_z dt$$

In order to calculate the integration constants which appear in these, it is necessary to know the vector $\bar{r}$ completely at one given instant.

If the displacement between instants $t = t_1$ and $t = t_2$ needs to be calculated, it may be done directly by means of the following definite integral:

$$\Delta\bar{r} = \bar{r}(t_2) - \bar{r}(t_1) = \int_{t_1}^{t_2} \bar{v}dt \qquad 1.5(2)$$

The vector equation represents **three** equations, one for each component. That for the x-component is as follows:

$$\Delta x = x(t_2) - x(t_1) = \int_{t_1}^{t_2} v_x \, dt$$

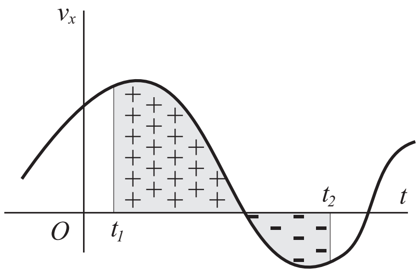


Fig.1.5-1

As it is known from the theory of definite integrals, the quantity Δx is represented by the area between the curve $v_x = v_x(t)$ and the t-axis from the value $t = t_1$ to $t = t_2$. In this representation, the area above the t-axis is taken as positive and that below, negative. This is shown in Figure 1.5-1. The two opposite signs indicate displacements in opposite directions.

Similar relationships exist for Δy and Δz and they may also be represented by areas on the graphs for $v_y = v_y(t)$ and $v_z = v_z(t)$ respectively.

Example: The velocity of a point is known as a function of time and is given by the following vector equation:

$$\bar{v} = \bar{v}(t) = (2t - 3)\hat{x} + (4t + 2)\hat{y} + (6t - 1)\hat{z} \ \mathrm{m\,s^{-1}}$$

in which the time, t, is measured in seconds. The position of the point is known for $t = 3$ s, and is given by $\bar{r}(3) = 2\hat{x} + 25\hat{y} + 21\hat{z}$ m.

(a) Calculate the position of the point as a function of time.

$$\begin{aligned}
\bar{r} = \int \bar{v} dt \quad &= \left(\int (2t-3)dt\right)\hat{x} + \left(\int (4t+2)dt\right)\hat{y} + \left(\int (6t-1)dt\right)\hat{z} \\
&= (t^2 - 3t + k_x)\hat{x} + (2t^2 + 2t + k_y)\hat{y} + (3t^2 - t + k_z)\hat{z} \\
\text{But} \quad \bar{r}(3) &= (k_x)\hat{x} + (24 + k_y)\hat{y} + (24 + k_z)\hat{z} \qquad \text{calculated from above} \\
\text{and} \quad \bar{r}(3) &= 2\hat{x} + 25\hat{y} + 21\hat{z} \qquad \text{specified in problem}
\end{aligned}$$

from which the numerical values of k_x, k_y and k_z may be calculated as follows: $k_x = 2, k_y = 1$ and $k_z = -3$, and $\bar{r}$ is known in full.

$$\bar{r} = (t^2 - 3t + 2)\hat{x} + (2t^2 + 2t + 1)\hat{y} + (3t^2 - t - 3)\hat{z} \ \mathrm{m}$$

(b) Calculate the displacement between $t = 1$s and $t = 2$s.

This may be calculated in two different ways. From the answer of question (a), it follows directly that

$$\Delta\bar{r} = \bar{r}(2) - \bar{r}(1) = (0\hat{x} + 13\hat{y} + 7\hat{z}) - (0\hat{x} + 5\hat{y} - \hat{z}) = 8\hat{y} + 8\hat{z} \text{ m.}$$

It may also be calculated directly by means of a definite integral as was explained in Equation 1.5(2).

$$\begin{aligned}\Delta\bar{r} &= \int_1^2 [(2t-3)\hat{x} + (4t+2)\hat{y} + (6t-1)\hat{z}]dt \\ &= (t^2 - 3t)\hat{x} + (2t^2 + 2t)\hat{y} + (3t^2 - t)\hat{z}|_1^2 \\ &= (-2\hat{x} + 12\hat{y} + 10\hat{z}) - (-2\hat{x} + 4\hat{y} + 2\hat{z}) \\ &= 8\hat{y} + 8\hat{z} \text{ m}\end{aligned}$$

(c) Calculate the acceleration of the point.

To calculate the acceleration, the same procedure is followed as that in 1.5.1(g). The reader can verify the correctness of the following answer:

$$\bar{a} = 2\hat{x} + 4\hat{y} + 6\hat{z} \text{ m s}^{-2}$$

1.5.3 Third kind: The acceleration vector, $\bar{a} = \bar{a}(t)$, is known

By definition $\bar{a} = \dfrac{d\bar{v}}{dt}$ from which it follows that $\bar{v} = \displaystyle\int \bar{a}\, dt$ 1.5(3)

which is equivalent to the following three equations:

$$v_x = \int a_x\, dt, \qquad v_y = \int a_y\, dt, \qquad v_z = \int a_z\, dt$$

In order to calculate the integration constants which appear in these integrals, it will be necessary to know $\bar{v}$ completely at a given instant.

The change in velocity between $t = t_1$ and $t = t_2$ may be calculated directly by means of a definite integral as follows:

$$\Delta\bar{v} = \bar{v}(t_2) - \bar{v}(t_1) = \int_{t_1}^{t_2} \bar{a}dt \qquad 1.5(4)$$

This vector equation represents three relationships, one for each component. That for the x-component is as follows:

$$\Delta v_x = v_x(t_2) - v_x(t_1) = \int_{t_1}^{t_2} a_x\, dt$$

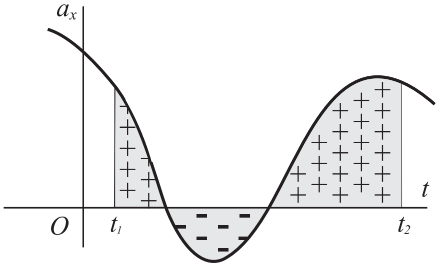


Fig.1.5-2

From the theory of definite integrals it is known that the quantity Δv_x may be represented as the area between the curve $a_x = a_x(t)$ and the t-axis from $t = t_1$ to $t = t_2$. This is shown in Figure 1.5-2. The area above the t-axis is positive and that below it, negative. The opposite signs represent changes in opposite directions.

Similar expressions exist for changes in the other two components of the velocity and they may also be represented by areas as is shown in Figure 1.5-2. The reader should take note of the perfect analogy that exists between the calculations in 1.5.3 and 1.5.2.

After the velocity vector has been calculated from the acceleration, the position vector may also be calculated, but it will be necessary to know the position vector, $\bar{r}$, completely at any given instant.

The complete solution of a problem of the kind discussed in 1.5.3, involves the calculation of the velocity and position vectors from the acceleration. For the calculation of the integration constants additional information is required. In the preceding discussion the velocity at a given instant and also the position at a given instant were used. It would also be possible to make the calculations if the positions were known at two different instants and this would make knowledge of the velocity vector at a given time superfluous. An example of this kind will be discussed in full.

Example: The acceleration of a point is constant, and is given by $\bar{a} = 4\hat{x}+7\hat{y}-4\hat{z}$ m s^{-2}. At time $t = 0$ the position is $\bar{r}(0) = 2\hat{x} - 2\hat{y} + \hat{z}$ m and at time $t = 1$ s, the velocity is $\bar{v}(1) = \hat{x} + 13\hat{y} - 2\hat{z}$ m s^{-1}.

(a) Calculate the velocity as a function of time.

$$\begin{aligned}
\bar{v} = \int \bar{a}\, dt &= \int (4\hat{x} + 7\hat{y} - 4\hat{z})dt \\
&= (4t + k_x)\hat{x} + (7t + k_y)\hat{y} + (-4t + k_z)\hat{z} \\
\text{but}\quad \bar{v}(1) &= (4 + k_x)\hat{x} + (7 + k_y)\hat{y} + (-4 + k_z)\hat{z} \qquad \text{(calculated)} \\
\text{and}\quad \bar{v}(1) &= \hat{x} + 13\hat{y} - 2\hat{z} \qquad \text{(given)}
\end{aligned}$$

from which the integration constants may be calculated. The values are as follows: $k_x = -3$, $k_y = 6$ and $k_z = 2$. The velocity vector, $\bar{v}(t)$, is now completely known.

$$\bar{v}(t) = (4t-3)\hat{x} + (7t+6)\hat{y} + (-4t+2)\hat{z} \quad \mathrm{m\,s^{-1}}$$

(b) Calculate the change in velocity between $t = 1$ s and $t = 2$ s.

$$\begin{aligned} \Delta\bar{v} = \bar{v}(2) - \bar{v}(1) &= (5\hat{x} + 20\hat{y} - 6\hat{z}) - (\hat{x} + 13\hat{y} - 2\hat{z}) \\ &= 4\hat{x} + 7\hat{y} - 4\hat{z} \ \mathrm{m\,s^{-1}} \end{aligned}$$

It may also be calculated directly by using a definite integral as explained in 1.5(4).

$$\begin{aligned} \Delta\bar{v} &= \int_1^2 (4\hat{x} + 7\hat{y} - 4\hat{z})dt = (4t)\hat{x} + (7t)\hat{y} + (-4t)\hat{z}|_1^2 \\ &= (8\hat{x} + 14\hat{y} - 8\hat{z}) - (4\hat{x} + 7\hat{y} - 4\hat{z}) \\ &= 4\hat{x} + 7\hat{y} - 4\hat{z} \ \mathrm{m\,s^{-1}} \end{aligned}$$

(c) Calculate the position of the point as a function of time.

$$\begin{aligned} \bar{r} &= \int \bar{v}\,dt = \int [(4t-3)\hat{x} + (7t+6)\hat{y} + (-4t+2)\hat{z}]\,dt \\ &= (2t^2 - 3t + 2)\hat{x} + (3,5t^2 + 6t - 2)\hat{y} + (-2t^2 + 2t + 1)\hat{z} \ \mathrm{m} \\ \text{since } \bar{r}(0) &= 2\hat{x} - 2\hat{y} + \hat{z} \ \mathrm{m} \end{aligned}$$

Example: The acceleration of a point is constant and given by $\bar{a} = 4\hat{x} - 6\hat{y} + 2\hat{z}$ $\mathrm{m\,s^{-2}}$. Its position at time $t = 1$ s is $\bar{r}(1) = 3\hat{x} - 2\hat{y} + 2\hat{z}$ m, and that at time $t = 2$ s, $\bar{r}(2) = 6\hat{x} - 9\hat{y} + 3\hat{z}$ m. Calculate its velocity and position as functions of time.

$$\begin{aligned} \bar{v} &= \int \bar{a}\,dt = \int (4\hat{x} - 6\hat{y} + 2\hat{z})\,dt \\ &= (4t + k_x)\hat{x} + (-6t + k_y)\hat{y} + (2t + k_z)\hat{z} \end{aligned}$$

It is impossible to calculate the integration constants k_x, k_y and k_z at this stage because no value for the velocity is known at a given time. The calculation may, however, proceed as if they were known and they will be calculated at a later stage.

$$\begin{aligned} \bar{r} &= \int \bar{v}\,dt = \int [(4t + k_x)\hat{x} + (-6t + k_y)\hat{y} + (2t + k_z)\hat{z}]dt \\ &= (2t^2 + k_x t + c_x)\hat{x} + (-3t^2 + k_y t + c_y)\hat{y} + (t^2 + k_z t + c_z)\hat{z} \\ \text{but } \bar{r}(1) &= (2 + k_x + c_x)\hat{x} + (-3 + k_y + c_y)\hat{y} + (1 + k_z + c_z)\hat{z} \quad \text{(calculated)} \\ \text{and } \bar{r}(1) &= 3\hat{x} - 2\hat{y} + 2\hat{z} \quad \text{(given)} \end{aligned}$$

By equating the corresponding components in the previous two expressions, the following three equations are obtained:

$$k_x + c_x = 1 \cdots (1) \qquad k_y + c_y = 1 \cdots (2) \qquad k_z + c_z = 1 \cdots (3)$$

$$\begin{array}{llll} \text{also} & \bar{r}(2) = & (8 + 2k_x + c_x)\hat{x} + (-12 + 2k_y + c_y)\hat{y} + (4 + 2k_z + c_z)\hat{z} & \text{(calculated)} \\ \text{and} & \bar{r}(2) = & 6\hat{x} - 9\hat{y} + 3\hat{z} & \text{(given)} \end{array}$$

This leads to the following three equations:

$$2k_x + c_x = -2 \cdots (4) \qquad 2k_y + c_y = 3 \cdots (5) \qquad 2k_z + c_z = -1 \cdots (6)$$

From these six equations the unknown constants may be calculated. Their values are as follows:

$$\begin{array}{lll} k_x = -3, & k_y = 2, & k_z = -2 \\ c_x = 4, & c_y = -1, & c_z = 3 \end{array}$$

The position and velocity vectors are now fully known as functions of the time.

$$\begin{aligned} \bar{r} &= (2t^2 - 3t + 4)\hat{x} + (-3t^2 + 2t - 1)\hat{y} + (t^2 - 2t + 3)\hat{z} \ \text{m} \\ \bar{v} &= (4t - 3)\hat{x} + (-6t + 2)\hat{y} + (2t - 2)\hat{z} \ \text{m s}^{-1} \end{aligned}$$

With these two vector functions known, all questions of the kind treated in 1.5.1 may be answered.

A strategy for the solution of problems in kinematics which involve time.

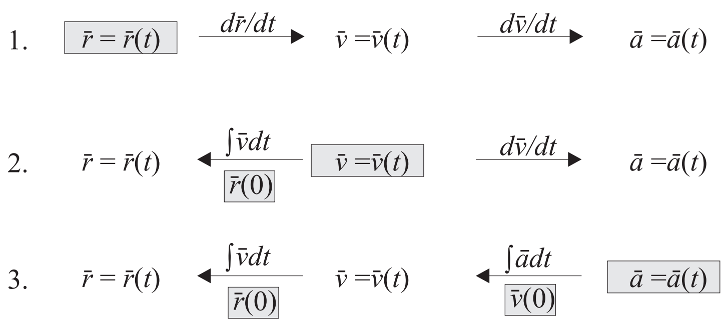
1. Read the question properly and make sure that it is understood what is given and what is to be calculated.
2. If possible, a sketch should be made on which all given quantities should be displayed.
3. If a frame of reference is not given, one must be chosen. Indicate it in the sketch. This includes the choice of $t = 0$ (the instant when the stop-watch is started).
4. Write down all the given quantities in terms of the chosen frame of reference.

5. Calculate those of $\bar{r} = \bar{r}(t)$, $\bar{v} = \bar{v}(t)$ and $\bar{a} = \bar{a}(t)$ which are not given. A schematic representation of the methods to be followed, is shown in 1.5.4.

6. The answers to questions in which the above are required, usually follow by the simple application of common sense.

1.5.4 A schematic representation of methods for solving problems in kinematics which involve time

If the quantities shown in boxes are known, the solution of a problem proceeds as indicated in the following schematic representation:



Although it is shown in the scheme that $\bar{r}(0)$ and $\bar{v}(0)$ should be known for the calculation of the integration constants, known values for $\bar{r}$ and $\bar{v}$ at *any* time will satisfy this need as was indicated in the examples.

1.6 The use of position as independent variable

Many problems in kinematics involve questions such as, "What is the position, velocity or speed of an object *at a given instant*?" Some problems do not involve the time at all and it might be required to answer questions such as, "What is the velocity or speed of an object when it is at a given position?" "Where will an object be when its speed has a given value?" or "What is the speed or velocity of an object as a function of a given position co-ordinate?" In answering questions like these, it is often possible to make use of time, but the calculations are usually long and involved. For this reason a technique is developed to deal with problems in which time is not used.

Consider the x-component of a motion.

By definition
$$a_x = \frac{dv_x}{dt}$$
$$= \frac{dv_x}{dx}\frac{dx}{dt} \qquad \text{(chain rule)}$$
$$= \frac{dv_x}{dx}v_x \qquad (v_x = dx/dt)$$

from which it follows
$$a_x\,dx = v_x\,dv_x \qquad 1.6(1)$$

The equation $a_x = (dv_x/dx)v_x$ is called a **differential equation** of the type **first order** (no higher-order derivatives than the first occur) and **first degree** (no derivative appears to a higher power than one). The variables in such an equation are always **separable**, i.e. they may be separated to appear on different sides of the equals sign. Equation 1.6(1) shows this equation after the separation of the variables has been completed. It is now ready to be solved by integration on both sides.

Before integration can take place, subsidiary information is required to calculate the integration constants. The supplementary information is known as the **boundary conditions** or **initial conditions**.

Suppose the following are the boundary conditions of a problem under consideration: $v_x = v_1$ where $x = x_1$ and $v_x = v_2$ where $x = x_2$. The solution is obtained by using these values as limits and integrating Equation 1.6.(1) on both sides.

$$\int_{x_1}^{x_2} a_x\,dx = \int_{v_1}^{v_2} v_x\,dv_x \qquad 1.6(2)$$

in which any *one* of x_1, x_2, v_1, v_2 or a_x may be an unknown quantity, or the integral may be used to calculate the function $v_x = v_x(x)$, i.e. the velocity component as a function of the position along a given axis. The use of this integral will be illustrated in the following three examples.

Examples:

1. The velocity of a motor car is $20\hat{x}$ m s^{-1} when the driver applies the brakes to bring it to rest over a distance of 50 m along a straight line. Assume that the acceleration is constant while the brakes are in action. Chose the origin where the brakes are applied. Calculate the acceleration.

From 1.6(2) follows
$$\int_0^{50} a_x\,dx = \int_{20}^{0} v_x\,dx$$
$$a_x x|_0^{50} = \frac{1}{2}v_x^2|_{20}^{0}$$

$$\begin{aligned} 50\,a_x &= \frac{1}{2}(0)^2 - \frac{1}{2}(20)^2 \\ \text{from which follows} \qquad a_x &= -4 \\ \text{and} \qquad \bar{a} &= -4\hat{x}\ \mathrm{m\,s^{-2}} \end{aligned}$$

2. The magnitude of the acceleration due to gravitation is equal to 10 $\mathrm{m\,s^{-2}}$. An object at rest falls from a height of 20 m above ground level. Calculate the speed at which it hits the ground. Disregard frictional drag.

Choose a frame of reference with $\hat{z}$ vertically upwards and origin on the ground. Then $\bar{a} = -10\hat{z}\ \mathrm{m\,s^{-2}}$ and $a_z = -10\ \mathrm{m\,s^{-2}}$. As before

$$\begin{aligned} \int_{20}^{0} -10\,dz &= = \int_{0}^{v_z} v_z\,dv_z \\ -10\,z|_{20}^{0} &= \frac{1}{2}v_z^2|_0^{v_z} \\ 200 &= \frac{1}{2}v_z^2 \\ \text{and} \qquad v = |v_z| &= 20\ \mathrm{m\,s^{-1}} \end{aligned}$$

Since it is required to calculate the *speed*, only the positive root is used.

3. An object is projected vertically upwards at an initial velocity of $\bar{v}(0) = 40\hat{z}$ $\mathrm{m\,s^{-1}}$. Gravitational acceleration is $\bar{a} = -10\bar{z}\ \ \mathrm{m\,s^{-2}}$. Calculate the velocity of the object as a function of its height above the ground ($z = 0$).

$$\begin{aligned} \int_{0}^{z} -10\,dz &= \int_{40}^{v_z} v_z\,dv_z \\ -10\,z|_0^z &= \frac{1}{2}v_z^2|_{40}^{v_z} \\ \text{from which follows} \qquad v_z &= \pm(1600 - 20z)^{\frac{1}{2}} \\ \text{and} \qquad \bar{v} = v_z\hat{z} &= \pm(1600 - 20z)^{\frac{1}{2}}\hat{z}\ \mathrm{m\,s^{-1}} \end{aligned}$$

For each allowed value of the height ($20z \leq 1600$) two values exist for the velocity, one whilst it is ascending and the other when it is descending.

1.7 Projectiles

1.7.1 Gravitational acceleration

As a result of **gravitation** (mass attracts mass), all free-falling bodies above the surface of the Earth experience approximately equal accelerations in the absence of frictional drag. This acceleration is directed towards the **centre of mass** of the Earth and its magnitude is usually indicated by the symbol g. Near the surface of the Earth $\bar{g} \approx -10\hat{r}$ m s^{-2}. Here $\hat{r}$ is the unit vector pointing radially away from the centre of the Earth. At the position where a **gravimeter** (an instrument to measure g) is kept at the National Museum in Pretoria, the value of g is given as $9,7861616 \pm 0,0000001$ m s^{-2}. This is an average value. Gravitational acceleration will be discussed fully in chapter 2.

1.7.2 Projectiles. The solution of a problem

For the purpose of this study, a **projectile** will initially be defined as a moving object which is acted upon by gravity and atmospheric frictional drag alone. The presence of other forces will be considered at a later stage. Examples: A bullet after it has left the barrel of a rifle, a stone after it has left the hand of the person who threw it, a bomb which was dropped from a moving aircraft.

In this study frictional drag will initially be disregarded. This will cause descriptions which sometimes are in error. In spite of this approximation, the salient features of projectile motion can be successfully derived. With a more thorough knowledge of the mathematics which is required, the reader will in future also be able to solve problems in which friction cannot be disregarded.

A projectile is usually characterised by two quantities: its **initial speed** and its **elevation angle** (elevation in short). The elevation is the angle measured from the horizontal plane to the initial velocity vector. It is rather convenient (but not necessary) to make the following two choices regarding the frame of reference in which problems will be solved: (i) Choose the origin of the system where the projectile motion starts, with $\hat{x}$ horizontally in the forward direction and $\hat{y}$ vertically upwards. (ii) Choose $t = 0$ when the projectile is at the origin.

These choices have the effect that only elevations between $\pi/2$ (vertically upwards) and $-\pi/2$ (vertically downwards) will occur. The choices were made for the following two reasons: (1) The initial position will always be equal to the zero vector. (2) Due to the absence of a z-component, all projectile problems reduce to two-dimensional problems.

With the proposed choices made, the initial position vector (always the zero vector), the initial velocity (which is calculated from the initial speed and the elevation angle), and the acceleration vector (always a y-component only) are written down and the solution proceeds in the way shown at number 3 in the scheme in 1.5.4. The complete solution of a projectile problem is shown as illustration. It should be remembered that a problem on projectile motion involves no new principles and, in fact, represents a very simplified application of the theory of kinematics.

Example:

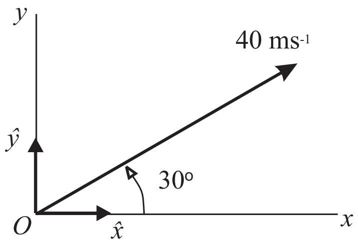


Fig. 1.7-1

An object is projected at an initial speed of 40 $\mathrm{m\,s^{-1}}$ and an elevation of 30°. $g = 10\ \mathrm{m\,s^{-2}}$. In accordance with the recommendations, the Cartesian system is chosen with its origin at the point where the motion begins, with $\hat{x}$ horizontally in the forward direction and $\hat{y}$ vertically upwards. Zero on the time scale, i.e. $t = 0$ is chosen at the instant when the object is at the origin. With this frame of reference which is shown in Figure 1.7-1, the values of $\bar{a}$, $\bar{v}(0)$ and $\bar{r}(0)$ are written down.

$$\begin{aligned}
\bar{r}(0) &= 0\hat{x} + 0\hat{y}\ \mathrm{m} \\
\bar{v}(0) &= (40\cos 30°)\hat{x} + (40\sin 30°)\hat{y} \\
&= 34,64\hat{x} + 20,00\hat{y}\ \mathrm{m\,s^{-1}} \\
\bar{a}(t) &= 0\hat{x} + (-10)\hat{y}\ \mathrm{m\,s^{-2}}
\end{aligned}$$

From this follows

$$\begin{aligned}
\bar{v}(t) = \int \bar{a}dt &= \int [0\hat{x} + (-10)\hat{y}]dt \\
&= k_x\hat{x} + (-10t + k_y)\hat{y} \\
&= 34,64\hat{x} + (-10t + 20,00)\hat{y}\ \mathrm{m\,s^{-1}}
\end{aligned}$$

also

$$\begin{aligned}
\bar{r}(t) = \int \bar{v}dt &= \int [34,64\hat{x} + (-10t + 20,00)\hat{y}]dt \\
&= (34,64t + c_x)\hat{x} + (-5t^2 + 20,00t + c_y)\hat{y} \\
&= (34,64t)\hat{x} + (-5t^2 + 20,00t)\hat{y}\ \mathrm{m}
\end{aligned}$$

The integration constants were calculated in each case by using the initial conditions, i.e. the values of $\bar{r}(0)$ and $\bar{v}(0)$.

The parametric equations of the space curve follow directly from the vector function $\bar{r} = \bar{r}(t)$:

$$x = x(t) = 34,64t\ \mathrm{m}, \qquad y = y(t) = -5t^2 + 20,00t\ \mathrm{m}$$

in which the time, t, is measured in seconds.

Solely because the choice of a frame of reference was such that the problem has been reduced to two dimensions, is it possible to eliminate the parameter t from the parametric equations to obtain a Cartesian equation for the space curve in the form, $y = y(x)$, as follows:

$$\begin{array}{llll} \text{From} & x & = & 34,64t, \qquad \text{it follows that} \qquad t = 2,887 \times 10^{-2}x \\ \text{and} & y & = & -5(2,887 \times 10^{-2}x)^2 + 20,00(2,887 \times 10^{-2}x) \\ & & = & -4,167 \times 10^{-3}x^2 + 5,774 \times 10^{-1}x \end{array}$$

This equation corresponds to the standard Cartesian equation for a parabola, viz. $y = ax^2 + bx + c$, in which $a < 0$ and $c = 0$. The space curve is thus a **convex parabola** (seen from above) which passes through the origin. The space curve of a projectile is often referred to as its **trajectory**.

The reader should take note that the steps which were followed, are those which were recommended for the solution of problems in 1.5.3. The reader is advised to compare 1.5.3 with what has been done. Up to this stage no thought has been given to any questions which might be asked; all the work done consists of the necessary calculations to provide the quantities which will enable one to answer questions. The following are known by calculation:

(1) The position vector of the projectile as a function of time

$$\bar{r} = (34,64t)\hat{x} + (-5t^2 + 20,00t)\hat{y} \text{ m} \qquad 1.7(1)$$

which is equivalent to

$$x = 34,64t \text{ m} \qquad y = -5t^2 + 20,00t \text{ m}$$

The last two equations are the parametric equations of the trajectory.

(2) The velocity vector as a function of time

$$\bar{v} = (34,64)\hat{x} + (-10t + 20,00)\hat{y} \text{ m s}^{-1} \qquad 1.7(2)$$

which is equivalent to

$$v_x = 34,64 \text{ m s}^{-1}, \qquad v_y = -10t + 20,00 \text{ m s}^{-1}$$

(3) The Cartesian equation for the space curve

$$y = -4,167 \times 10^{-3}x^2 + 5,774 \times 10^{-1}x \qquad 1.7(3)$$

By using this information about the projectile it will be possible to answer any question about its flight. This will be illustrated in the following examples. For a beginner the most difficult step is to give a mathematical interpretation to a question formulated in words. If this can be done successfully, the solution is usually very simple. The reader is advised to study this aspect with great care in the examples that follow.

(a) Calculate the time for the projectile to reach its maximum height.

The concept of "reaching maximum height" may be described mathematically in three different ways: (i) The vertical component of the velocity, v_y, is equal to zero. (ii) y (which is the height) is a maximum. (This allows the use of the theory of maxima and minima.) (iii) The slope of the trajectory, i.e. dy/dx, is equal to zero. The use of any one of these three mathematical statements will lead to the correct answer, but (i) is probably the simplest since it supplies the answer most directly.

If $v_y = 0$, the projectile is at maximum height. From 1.7(2):

$$v_y = -10t + 20,00 = 0 \qquad \text{from which follows} \qquad t = 2 \text{ s}$$

The possibility of reaching a maximum height is, of course, only applicable to projectiles with a positive elevation.

(b) Calculate the maximum height of the projectile.

The maximum height is, of course, the maximum value of y, the distance above the x-axis. The easiest way to calculate this, is by using the answer to the previous question. The maximum height is simply the value of $y(2)$. From 1.7(1) follows

$$y(2) = -5(2)^2 + 20(2) = 20 \text{ m}$$

(c) Calculate the time of flight over a horizontal field.

For this problem, the flight of the projectile ends when $y = 0$. From 1.7(1)

$$y = -5t^2 + 20,00t = 0 \qquad \text{from which follows} \qquad t = 0 \quad \text{or} \quad t = 4 \text{ s}$$

The projectile is at $y = 0$ at both these times but at $t = 0$ it was, by the initial choice, at the origin. The correct answer is thus $t = 4$ s.

If the object had been projected from a building with height 20 m, the flight would have ended when $y = -20$ m.

(d) Calculate the range of the projectile over a horizontal field.

To avoid confusion the **range** of a projectile over a horizontal field is defined as the value of the horizontal component of the position vector when the flight ends. This definition is only useful for relatively short flights. If the range of an intercontinental ballistic missile is to be measured, it will have to be measured along a **geodesic** (i.e. along the circle with radius equal to that of the earth and which contains the firing point and end-point of the trajectory – also called a **great circle**).

For this problem, the range is the value of x when $y = 0$ at time $t = 4$ s. From 1.7(1)

$$x(4) = 34,64 \times 4 = 138,6 \text{ m}$$

(e) Calculate the final velocity of the projectile.

The flight ends when $y = 0$ and $t = 4$ s. From 1.7(2)

$$\begin{aligned} \bar{v}(4) &= (34,64)\hat{x} + (-10 \times 4 + 20,00)\hat{y} \\ &= (34,64)\hat{x} + (-20,00)\hat{y} \text{ m s}^{-1} \end{aligned}$$

It is of importance to note that the horizontal component of the velocity remained unchanged throughout the entire flight (the acceleration does not have a component in the $\hat{x}$-direction), whilst that of the y-component changed. This flight ended with a y-component of the velocity equal in magnitude to that at the beginning, but with opposite sign. The velocity components for a projectile with positive elevation are shown at different positions along its trajectory in Figure 1.7-2.

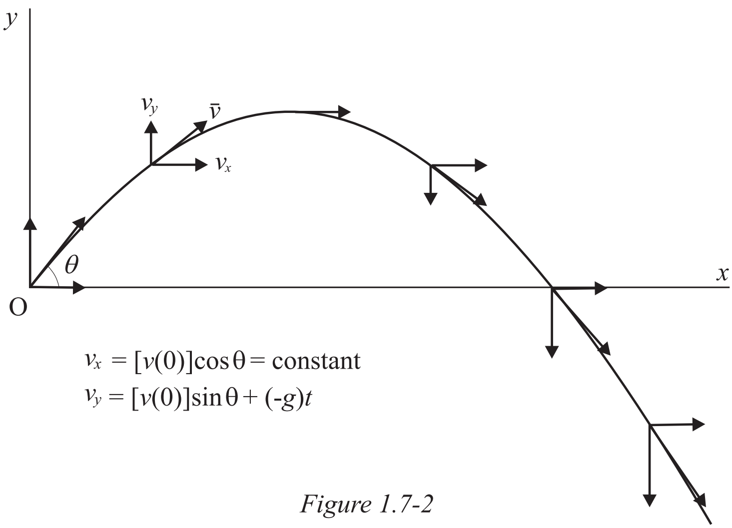


Figure 1.7-2

(f) Calculate the final speed of the projectile.

$$v(4) = |\bar{v}(4)| = [(34,64)^2 + (-20,00)^2]^{\frac{1}{2}} = 40,00 \ \mathrm{m\,s^{-1}}$$

In all the preceding problems use was made of Equations 1.7(1) and 1.7(2) which give the position and velocity vectors as functions of time. Some of these problems could have been solved directly making use of the Cartesian equation for the trajectory which is given in 1.7(3). This is illustrated in the following question.

(g) Use the equation for the trajectory to calculate the range of the projectile over a horizontal field.

The range is the value of x where $y = 0$. From 1.7(3)

$$(-4,167 \times 10^{-3})x^2 + (5,774 \times 10^{-1})x = 0$$

This equation has two roots: $x = 0$ and $x = 138,6$ m. The value $x = 0$ represents the initial position and therefore the second answer is the correct one. Compare this with the answer obtained for question (d).

(h) Calculate the angle of inclination of the trajectory.

The **angle of inclination** of the trajectory is defined as the angle measured from the forward horizontal direction to the velocity vector. When the projectile is ascending this angle is positive and when descending, it is negative. If the axes where chosen as recommended, this angle can only assume values between $\pi/2$ and $-\pi/2$. There are two methods to calculate it.

(i)

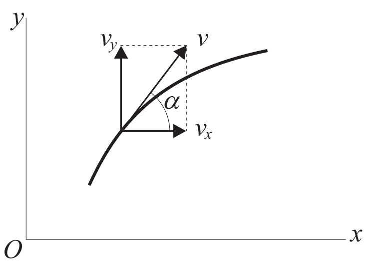


Figure 1.7-3

Let α represent the angle of inclination. From Figure 1.7-3 it can be seen that

$$\begin{aligned} \tan\alpha &= v_y/v_x \\ \alpha &= \arctan v_y/v_x \end{aligned}$$

While the projectile is ascending, both v_x and v_y are positive so that $0 \le \alpha \le \pi/2$ ($\tan\alpha$ is positive). While the projectile is descending, only v_y is negative and then $-\pi/2 \le \alpha \le 0$.

By using the velocity components in 1.7(2) it is now possible to calculate the inclination angle at any given time. It is, of course, possible to rewrite these components as functions of x by using the parametric equation for x. Then it will also be possible to calculate α for any given value of x.

(ii) From the theory of differentiation it is known that $\tan\alpha = dy/dx$. From 1.7(3)

$$\begin{aligned} y &= (-4,167 \times 10^{-3})x^2 + (5,774 \times 10^{-1})x && \text{so that} \\ \frac{dy}{dx} &= (-8,334 \times 10^{-3})x + (5,774 \times 10^{-1}) && \text{and} \\ \alpha &= \arctan\left[-8,334 \times 10^{-3}x + 5,774 \times 10^{-1}\right] \end{aligned}$$

which can be calculated if a given x-value is specified. If it is required to calculate α at a given time, the value of x may first be calculated from 1.7(1).

1.7.3 Range as a function of elevation

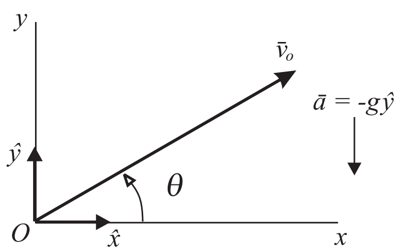


Fig. 1.7-4

In order to illustrate some salient features of projectile motion, the problem will be solved for the general case, using symbols rather than numerical values. The initial speed is v_0, the elevation θ and gravitational acceleration g. As was done previously the origin is chosen at the point where the motion commences, with $\hat{x}$ horizontally forwards and $\hat{y}$ vertically upwards. The beginning of the time scale is defined by $\bar{r}(0) = \bar{0}$.

In this frame of reference we have:

$$\begin{aligned} \bar{r}(0) &= 0\hat{x} + 0\hat{y} \\ \bar{v}(0) &= (v_0 \cos\theta)\hat{x} + (v_0 \sin\theta)\hat{y} \\ \bar{a} &= 0\hat{x} + (-g)\hat{y} \end{aligned}$$

The vectors $\bar{v}$ and $\bar{r}$ are now calculated in the same manner as was done in the numerical examples. For the calculation of the integration constants, the given values of $\bar{v}(0)$ and $\bar{r}(0)$ are used. The reader should verify the following results:

$$\bar{v}(t) = (v_0 \cos\theta)\hat{x} + (-gt + v_0 \sin\theta)\hat{y} \qquad 1.7(4)$$

$$\bar{r}(t) = (v_0 \cos\theta)t\hat{x} + [-\frac{1}{2}gt^2 + (v_0 \sin\theta)t]\hat{y} \qquad 1.7(5)$$

If the projectile is launched over a horizontal field, the flight will end when $y = 0$. The time of flight, T, may be calculated by equating y to zero in 1.7(5).

$$\text{time of flight} = T = (2v_0/g)\sin\theta$$

The range, X, is the value of x when $t = T$, i.e. the value of x when the flight ends. From 1.7(5) follows:

$$\begin{aligned} \text{range} &= X = (v_0 \cos\theta)T = (v_0 cos\theta)(2v_0/g)\sin\theta \\ &= (v_0^2/g)(2\sin\theta\cos\theta) \\ &= (v_0^2/g)\sin 2\theta \end{aligned} \qquad 1.7(6)$$

Equation 1.7(6) expresses $X = X(\theta)$ from which the range, X, may be calculated for any given value of the elevation, θ. From the theory of maxima and minima we know that $dX/d\theta = 0$ at each relative extreme value. From 1.7(6) follows

$$\frac{dX}{d\theta} = (2v_0^2/g)\cos 2\theta$$

which is equal to zero if $\cos 2\theta = 0$. From this it follows that X could have relative extreme values if $2\theta = 3\pi/2$ or $\pi/2$. Only values of 2θ between 0 and π are of practical interest. This is because $\hat{x}$ was chosen in the forward direction, and for a flight over a horizontal field θ can only assume values between 0 and $\pi/2$. The reader can verify by examination of the sign of $d^2X/d\theta^2$, that the first value of 2θ gives a minimum value for X, and the second, a maximum. The range is thus a maximum for an elevation of $\theta = \pi/4 = 45°$. The minimum value has the same magnitude, but opposite sign and corresponds to a projectile moving in the opposite direction. According to the choice of axes this is not a possible solution.

It is of interest to note that elevations $\theta = \pi/4 - \alpha$ and $\theta = \pi/4 + \alpha$ give the same range for a specified initial speed. If these two angles are used in Equation 1.7(6), the following is found:

$$\begin{aligned} \sin 2(\pi/4 - \alpha) &= \sin(\pi/2 - 2\alpha) \\ &= \cos 2\alpha \\ \text{and} \qquad \sin 2(\pi/4 + \alpha) &= \sin(\pi/2 + 2\alpha) \\ &= \cos 2\alpha \end{aligned}$$

from which it can be seen that a given range, excluding the maximum, resulting from the same initial speed, can be reached by a high trajectory and also by a low trajectory. This fact has been known to artillerists for many centuries and is applied according to the existing needs. The high trajectory has a longer time of flight and is thus more susceptible to the influences of frictional drag. The lower trajectory is always used with small arms. The mortar is an example of a weapon with which the high trajectory is used exclusively.

Two trajectories with the same initial speed but different elevations are shown in Figure 1.7-5. Because the elevations are symmetrical about $\pi/4$, the ranges are the same. The trajectory with maximum range is also shown in the sketch.

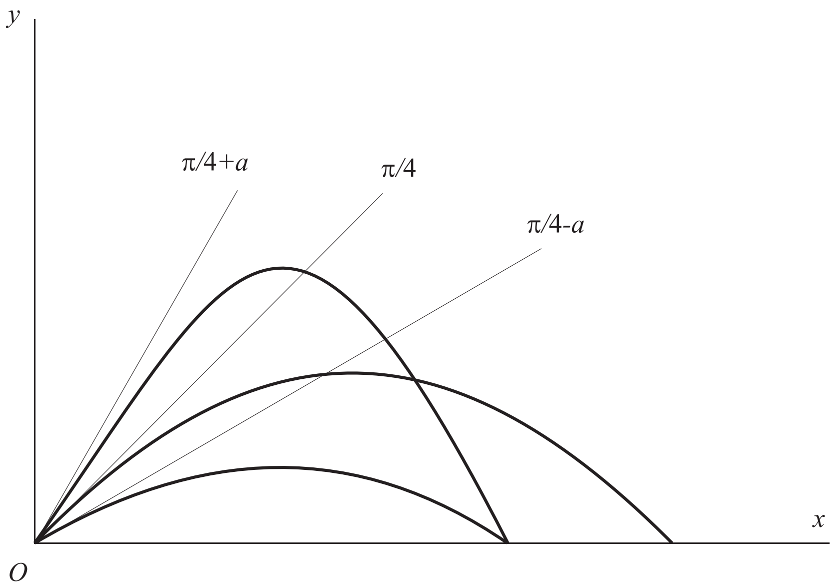


Figure 1.7-5

1.8 Relativistic kinematics

If two observers are in motion relative to each other, the kinematic quantities (position, speed, velocity and acceleration) by means of which one of them describes a motion, will in general differ from those used by the other. A good example is a person walking along the platform of railway station. His motion as seen by an observer standing on the platform, is quite different from that seen by an observer in a moving train. Although the two observers use different quantities to describe the motion of the walking person, the relationship between those quantities is relatively simple. The physics which deals with such relationships, is known as **relativistic kinematics**.

1.8.1 The Galilei transformation for position

In honour of the eminent scientist Galileo Galilei (1564 - 1642) who laid the foundations for kinematics on which Isaac Newton built, **classical relativity** is described by a set of equations known as the **Galilei transformation**.

Consider two frames of reference which we will name S and S'. Co-ordinates and time measured relative to S, are indicated by (x, y, z, t) and those measured relative to S', by (x', y', z', t'). For initial considerations, the two frames of reference will have to comply with the following:

(a) They will both be **inertial systems**. An inertial system is one in which a mass will only accelerate when acted upon by a force. Rotating and accelerating frames of reference are not inertial systems.
(b)Their x-axes coincide. Their y-axes and z-axes are parallel.
(c) The beginning of their time scales is defined by $t = t' = 0$ when their origins coincide.
(d) The origin of S' moves at **constant velocity** $\bar{u} = u\hat{x}$ relative to S.

NB It is of prime importance to remember that the two frames of reference are chosen to comply with these conditions. This was a prerequisite for the derivation of the Galilei transformation. If a relativity problem has to be solved, the axes are simply chosen in this way. In 1.8.4 it will be shown that the equations will be valid for any constant relative velocity, $\bar{u}$.

Figure 1.8-1 shows such a pair of frames of reference. The distance between the planes $x = 0$ and $x' = 0$ depends on the time and is given by $ut = ut'$ as shown in the sketch. Consider a point P with co-ordinates (x, y, z) relative to S and (x', y', z') relative to S'.

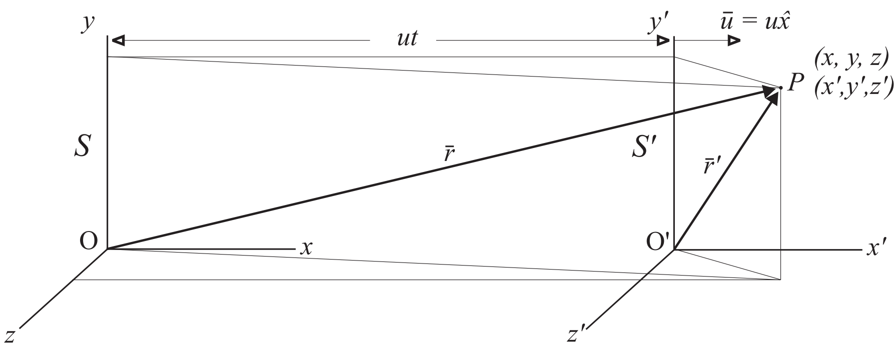


Figure 1.8-1

From the sketch it can be seen that

$$\begin{aligned} x &= x' + ut \\ y &= y' \end{aligned} \quad \begin{matrix} \text{or} \\ \text{and} \end{matrix} \quad \begin{aligned} x' &= x - ut \\ z &= z' \end{aligned}$$

Everyday experience shows that the time scales on two identical clocks, are apparently the same and it is not necessary to make a distinction between t and t'. The following set of equations gives the co-ordinates of P relative to S in terms of those relative to S'. They are known as the **Galilei transformation**.

$$\begin{aligned} x &= x' + ut \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned} \qquad \text{i.e.} \qquad \begin{aligned} \bar{r} &= \bar{r}' + \bar{u}t \\ t &= t' \end{aligned} \qquad 1.8(1)$$

The subjects of the equations in 1.8(1) may be changed to give a set of equations which express the co-ordinates of P relative to S' in terms of those in S. This set is known as the **inverse Galilei transformation**.

$$\begin{aligned} x' &= x - ut \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \qquad \text{i.e.} \qquad \begin{aligned} \bar{r}' &= \bar{r} - \bar{u}t \\ t' &= t \end{aligned} \qquad 1.8(2)$$

In this transformation, t and t' are interchangeable in any calculation because time apparently progresses equally in both frames of reference.

1.8.2 The Galilei transformation for velocities

If point P moves relative to S at velocity $\bar{v} = d\bar{r}/dt$, it moves relative to S' at velocity $\bar{v}' = d\bar{r}'/dt' = d\bar{r}'/dt$. From 1.8(1) follows:

$$\frac{dx}{dt} = \frac{dx'}{dt} + u \qquad \text{(differentiate} \quad x = x' + ut)$$

so that

$$\begin{aligned} v_x &= v'_x + u \\ v_y &= v'_y \\ v_z &= v'_z \end{aligned} \qquad \text{i.e.} \qquad \bar{v} = \bar{v}' + \bar{u} \qquad 1.8(3)$$

The inverse transformation which follows from 1.8(3) is:

$$\begin{aligned} v'_x &= v_x - u \\ v'_y &= v_y \\ v'_z &= v_z \end{aligned} \qquad \text{i.e.} \qquad \bar{v}' = \bar{v} - \bar{u} \qquad 1.8(4)$$

1.8.3 The Galilei transformation for accelerations

If point P accelerates relative tot S and the acceleration is given by $\bar{a} = d\bar{v}/dt$, its acceleration relative to S' is given by $\bar{a}' = d\bar{v}'/dt$. From 1.8(3) follows:

$$\begin{aligned} a_x &= a'_x \\ a_y &= a'_y \\ a_z &= a'_z \end{aligned} \qquad \text{i.e.} \qquad \bar{a} = \bar{a}' \qquad 1.8(5)$$

Acceleration is the same in both frames of reference. Acceleration is said to be **invariant** under a Galilei transformation (**Galilei invariant** in short).

1.8.4 The Galilei transformation for a relative velocity not parallel to $\hat{x}$

In the preceding work only examples where the relative velocity of the two frames of reference was parallel to their common x-axis (i.e. $\bar{u} = u\hat{x}$) were considered and the result was that only the x-components of position and velocity were affected. Now consider the same two frames of reference (corresponding axes parallel and the origins coincide when $t = t' = 0$) but with relative velocity $\bar{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}$.

In the same manner as that which was used previously, it can be shown that the y-component of the relative velocity will have an effect only on the y-components of the position, and velocity vectors. It can also be shown that the effect is identical to that which was deduced for the x-components in the preceding sections. The same applies to the z-components and it is generally true that

$$\begin{aligned} \bar{r} &= \bar{r}' + ut \\ \bar{v} &= \bar{v}' + \bar{u} \\ \bar{a} &= \bar{a}' \\ t &= t' \end{aligned} \quad \text{with inverse transformation} \quad \begin{aligned} \bar{r}' &= \bar{r} - ut \\ \bar{v}' &= \bar{v} - \bar{u} \\ \bar{a}' &= \bar{a} \\ t' &= t \end{aligned} \qquad 1.8(6)$$

irrespective of the direction of the constant relative velocity, $\bar{u}$.

1.8.5 Problems in which the Galilei transformation is used

In all relativity problems there have to be **two** frames of reference (which may be named S and S') and a point or an object of which the position, velocity and acceleration are measured from both systems. It is not important which system is named S and which is named S', but once the choice has been made, it is of prime importance to remember that the velocity of the origin of S' relative to S, is called $\bar{u}$. Position, velocity and acceleration relative to S are indicated by $\bar{r}, \bar{v}$ and $\bar{a}$ respectively, and that relative to S', $\bar{r}', \bar{v}'$ and $\bar{a}'$ respectively. With this choice made, it only remains to determine what is given and what is to be calculated. The calculations are made by means of the Galilei transformations. The two chosen frames of reference must, however, comply with the conditions (a), (b) and (c) mentioned in 1.8.1. In 1.8.4 it was shown that the Galilei transformation is valid for any constant relative velocity, $\bar{u}$.

Examples:

1. An aircraft has a velocity of 100 m s^1 due north relative to the ground. A child in the plane throws a ball at a speed of 4 m s^{-1}. Calculate the velocity and speed of the ball relative to the ground if the direction in which the child throws it, is (a) north, (b) south, (c) west. Choose S in rest relative to the ground and S' in rest relative to the aircraft with $\hat{x}$ north and $\hat{y}$ west. The relative velocity, $\bar{u}$, is the velocity of the plane relative to the ground, i.e. the velocity of S' relative to S.

With this choice, $\bar{u} = 100\hat{x}\ \text{m s}^{-1}$. The velocity of the ball is specified in each case relative to S' (according to the choice which was made) and for this reason it has to be indicated by $\bar{v}'$ and not $\bar{v}$. In each case the velocity has to be calculated relative to the ground (S according to the choice), and therefore the unknown velocity is indicated by $\bar{v}$.

(a) $\bar{v}' = 4\hat{x}\ \text{m s}^{-1}$ and $\bar{u} = 100\hat{x}\ \text{m s}^{-1}$.

According to the Galilei transformation

$$\begin{aligned} \bar{v} &= \bar{v}' + \bar{u} = 4\hat{x} + 100\hat{x} = 104\hat{x} \ \mathrm{m\,s^{-1}} \\ v &= |\bar{v}| = 104 \ \mathrm{m\,s^{-1}} \end{aligned}$$

(b) $\bar{v}' = -4\hat{x}\ \mathrm{m\,s^{-1}}$ and $\bar{u} = 100\bar{x}\ \mathrm{m\,s^{-1}}$.

As before

$$\begin{aligned} \bar{v} &= \bar{v}' + \bar{u} = -4\hat{x} + 100\hat{x} = -96\hat{x} \ \mathrm{m\,s^{-1}} \\ v &= |\bar{v}| = 96 \ \mathrm{m\,s^{-1}} \end{aligned}$$

(c) $\bar{v}' = 4\hat{y}\ \mathrm{m\,s^{-1}}$ and $\bar{u} = 100\hat{x}\ \mathrm{m\,s^{-1}}$.

And once again

$$\begin{aligned} \bar{v} &= \bar{v}' + \bar{u} = 100\hat{x} + 4\hat{y} \ \mathrm{m\,s^{-1}} \\ v &= |\bar{v}| = [(100)^2 + (4)^2]^{\frac{1}{2}} = 100,08 \ \mathrm{m\,s^{-1}} \end{aligned}$$

If the aircraft had been chosen to be S and the Earth, S', then $\bar{u} = -100\hat{x}$ $\mathrm{m\,s^{-1}}$. The velocity of the ball relative to the plane, would have to be called $\bar{v}$, and the unknown velocity relative to the ground, $\bar{v}'$. Application of the Galilei transformation leads to exactly the same results. It is recommended that the reader work through the problem with the latter choice.

2. At time $t = t' = 0$, ship B is 12 km north-east of ship A. A sails at 15 $\mathrm{m\,s^{-1}}$ due north, and B at 10 $\mathrm{m\,s^{-1}}$ in direction W 30°N. (a) Calculate the velocity of B relative to A. (b) Calculate the speed and direction of B relative to A and show that they are not on a collision course. (c) Calculate the shortest distance at which they pass each other. (d) Calculate the time interval from the initial conditions until this closest distance is reached. (e) If everything else remains the same, calculate the speed that B must assume to ensure a rendezvous course.

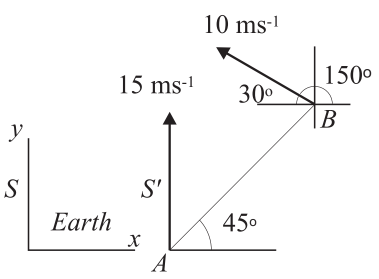


Figure 1.8-2

Choose the Earth as S and ship A, S'. This is shown is Figure 1.8-2. With Earth represented by S, $\hat{y}$ is chosen north and $\hat{x}$ east. Another choice would have worked as well. Now $\bar{u} = 15\hat{y}$ $\mathrm{m\,s^{-1}}$. The velocity of ship B is known relative to the Earth (S) and *must* be indicated by $\bar{v}$ (*not* $\bar{v}'$). In the chosen frame of reference we have:

$$\begin{aligned} \bar{v} &= (10\cos 150^\circ)\hat{x} + (10\sin 150^\circ)\hat{y} \\ &= -8,660\hat{x} + 5,000\hat{y} \ \mathrm{m\,s^{-1}} \end{aligned}$$

(a) According to the Galilei-transformation we have:

$$\begin{aligned}\bar{v}' = \bar{v} - \bar{u} &= (-8,660\hat{x} + 5,000\hat{y}) - (15,000)\hat{y} \\ &= -8,660\hat{x} - 10,000\hat{y} \text{ m s}^{-1}\end{aligned}$$

(b) The relative velocity and direction can be calculated as follows:

$$v' = |\bar{v}'| = [(-8,660)^2 + (-10,000)^2]^{\frac{1}{2}} = 13,23 \text{ m s}^{-1}$$

$$\text{and} \qquad \tan\theta = \frac{-10,000}{-8,660} \qquad \text{so that} \qquad \theta = -130,9^\circ$$

y It is, of course, much easier to use the function on a calculator for transforming directly from rectangular co-ordinates to polar co-ordinates.

To the observer on A it appears that B has a speed of 13,23 m s^{-1} in direction S $40,9^\circ$W. If the angle had been 45°, the ships would have been on a collision course. This is shown in Figure 1.8-3(a).

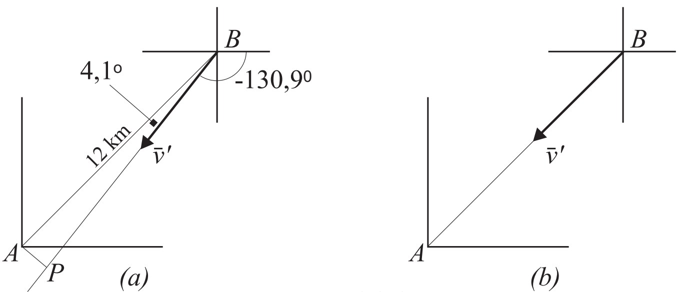


Figure 1.8-3

(c) The shortest distance between the two ships is given by the perpendicular from A to the apparent course of B. This is AP in Figure 1.8-3(a). The angle ABP is equal to $4,1^\circ$ and since $\sin A\hat{B}P = AP/AB = AP/12$, we have that $AP = 12\sin 4,1^\circ = 0,8580$ km.

(d) The apparent distance travelled by B until the closest position is reached, is BP which is given by $AB\cos 4,1^\circ = 12\cos 4,1^\circ = 11,97$ km $= 11,97 \times 10^3$ m. The time taken to reach this position is given by $\Delta t = BP/v' = (11,97 \times 10^3)/13,23 = 904,7$ s $=$ 15,08 minutes.

(e) For a rendezvous course the speed of B has to be changed. Let this be represented by v. The velocity relative to the Earth is then:

$$\begin{aligned}\bar{v} &= (v\cos 150^\circ)\hat{x} + (v\sin 150^\circ)\hat{y} \\ &= -0,8660\, v\hat{x} + 0,5000\, v\hat{y} \text{ m s}^{-1}\end{aligned}$$

From the Galilei transformation we have: $\bar{v}' = \bar{v} - \bar{u}$, so that

$$\bar{v}' = (-0,8660\,v)\hat{x} + (0,5000\,v - 15,0000)\hat{y} \ \ \mathrm{m\,s^{-1}}$$

If this velocity makes an angle of -135° with the x-axis as shown in Figure 1.8-3(b), the ships will be on a rendezvous course. To satisfy this condition we have:

$$\tan(-135°) = \frac{0,5000\,v - 15,0000}{-0,8660\,v} = 1$$

If this equation is solved for v, the value $v = 10,98\ \mathrm{m\,s^{-1}}$ is obtained.

3. The air speed of a light aircraft is 200 $\mathrm{km\,h^{-1}}$. The pilot wishes to reach a destination which is 375 km due east. A constant wind of 20 $\mathrm{km\,h^{-1}}$ in direction S 10°W prevails along the entire course. (a) Calculate the compass bearing by which the destination will be reached. (b) Calculate the ground speed and the flight time.

Air speed refers to the speed of the aircraft relative to the air. This is what a pilot reads on the air-speed indicator and it is not influenced by wind. We choose S at rest relative to the ground and S' at rest relative to the air. According to this choice, the velocity of the air has to be represented by $\bar{u}$. The pilot reads the velocity of his aircraft relative to the air from his instruments (air-speed indicator and compass) and it has to be represented by $\bar{v}'$ in the chosen frame of reference. The velocity of the aircraft relative to the ground is $\bar{v}$. The frames of reference are shown in Figure 1.8-4.

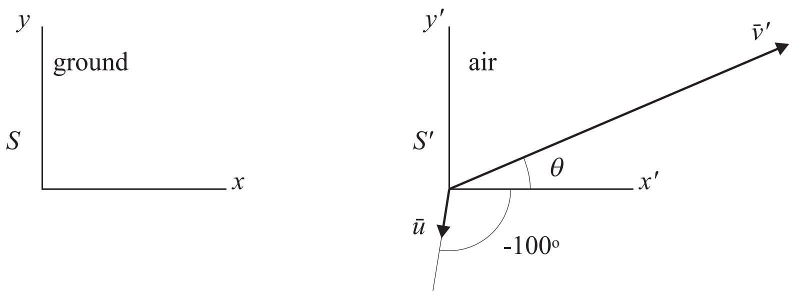


Figure 1.8-4

In accordance with the choice of frames of reference, we have:

$$\begin{aligned}\bar{u} &= (20\cos[-100°])\,\hat{x} + (20\sin[-100°])\,\hat{y} \\ &= (-3,473)\hat{x} + (-19,696)\hat{y}\ \ \mathrm{km\,h^{-1}}\end{aligned}$$

(a) Let the compass reading be E θ°N. Then

$$\bar{v}' = (200\cos\theta)\hat{x} + (-200\sin\theta)\hat{y}\ \ \mathrm{km\,h^{-1}}$$

From the Galilei transformation it follows that:

$$\begin{aligned}\bar{v} &= \bar{v}' + \bar{u} \\ &= (200\cos\theta - 3,473)\hat{x} + (200\sin\theta - 19,696)\hat{y} \ \text{km}\,\text{h}^{-1}\end{aligned}$$

If the aircraft is to fly due east, then $v_y = 0$, i.e. $200\sin\theta = 19,696$. From this follows:

$$\theta = \arcsin 0,0985 = 5,7^\circ$$

The bearing (compass reading) should be E 5,7°N, or 84,3° in the notation used by navigators.

(b) Now $v = |\bar{v}| = |200\cos 5,7^\circ - 3,473| = 195,5 \ \text{km}\,\text{h}^{-1}$.
The flight time is given by:

$$\Delta t = 375/195,5 = 1,92 \ \text{h}$$

1.8.6 The Lorentz transformation for position and time

By using an extremely sensitive instrument known as a **Michelson interferometer**, the two physicists, Michelson and Morley (1887), discovered a remarkable fact: The speed of light in a vacuum ($c \approx 3 \times 10^8 \ \text{m}\,\text{s}^{-1}$) is *independent* of the relative velocity of the source of the light and the observer who measures it.

If this observation is compared to the results of classical relativity, it means that the speed of light plus or minus any other speed should be equal to the speed of light. This is definitely in conflict with the Galilei transformation. People like Fitzgerald, Larmor, Lorentz and others, tried to explain the strange observation within the framework of classical physics. Lorentz succeeded in deriving a *Lorentz-transformation* by means of which the "negative result" of Michelson and Morley could be explained.

Albert Einstein (1879 - 1955) worked from the assumption that the observations of Michelson and Morley were correct. In 1905 he published his special theory of relativity which is based on two postulates (i) No experiment exists by which absolute motion or absolute rest of an inertial system can be determined; only the relative motion of two such systems at a constant velocity has any meaning. A different way of saying this is that the laws of physics are the same in all inertial systems.(ii) The speed of light in a vacuum as measured by an observer in an inertial system, is independent of the relative motion at constant velocity of the source of the light. In his theory Einstein showed that the Lorentz transformation is a necessary result of these postulates.

The Lorentz transformation will not be derived in this text. It will only be stated, explained and used to derive a number of interesting results. It will also be shown that at relative speeds which are low compared to that of light, the Lorentz and Galilei transformations, for all practical purposes, lead to the same results.

The Lorentz transformation which follows, is based upon the two frames of reference which are described in 1.8.1 and shown in Figure 1.8-1. They comply with the conditions (a) to (d). In this study only cases in which the relative velocity is parallel to the common x-axis will be considered. No extension will be made for a relative velocity with three components as was done in the case of the Galilei transformation.

For these two frames of reference, the Lorentz transformation is as follows:

$$\begin{aligned} x &= \gamma(x' + ut') \\ y &= y' \\ z &= z' \\ t &= \gamma(t' + ux'/c^2) \end{aligned} \quad \text{with inverse transformation} \quad \begin{aligned} x' &= \gamma(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - ux/c^2) \end{aligned} \qquad 1.8(7)$$

in which c is the speed of light in free space (vacuum), and γ the **Lorentz constant** for the two frames of reference which is defined by the following:

$$\gamma = (1 - u^2/c^2)^{-\frac{1}{2}} \qquad 1.8(8)$$

In order to form a notion of the manner in which the relative speed influences the magnitude of this constant, the value of γ is calculated for a number of

u	$\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$
$0,0100c$	1,000050003
$0,1000c$	1,005037815
$0,5000c$	1,154700538
$0,6000c$	1,250000000
$0,8000c$	1,666666667
$0,9000c$	2,294157338
$0,9999c$	70,712445950

Table 1.8-1

values of u and shown in table 1.8-1. In Figure 1.8-5 a graphical representation of the function $\gamma = \gamma(u)$ is shown. In the table and on the graph, the relative speed, u, is expressed as fractions of the speed of light in free space, c.

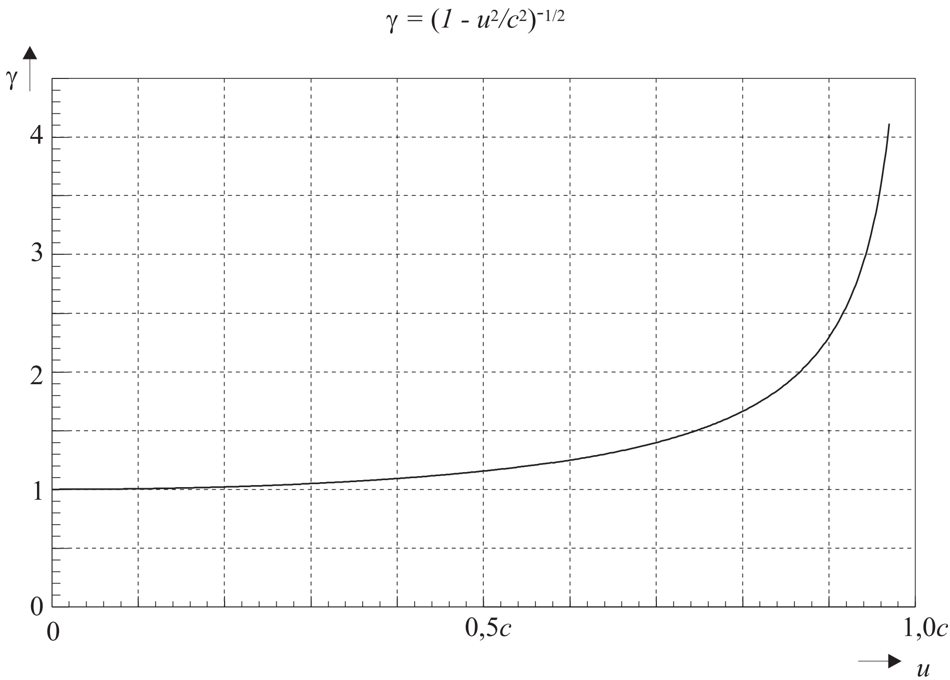


Figure 1.8-5

The quantity γ has a number of important properties which play a role in all problems on relativity: (1) If u is much smaller than c, the value of γ does not differ much from 1. The implications are that the results of the Galilei and Lorentz transformations at low relative speed, are practically indistinguishable. (2) As the value of u approaches that of c, γ approaches infinity. (3) If u exceeds c, γ is imaginary. In such cases the answers of the Lorentz transformation would also be imaginary. But the Lorentz transformation always deals with real quantities such as the components of position vectors and points in time. This implies that an upper limit exists for u.

Consider the Lorentz transformation given in 1.8(7) once again for a case in which u is much less than c. It would be a good approximation to equate γ to one and the ratio u/c^2 to zero. The Lorentz and Galilei transformations are practically the same under these conditions.

A surprising result of the Lorentz transformation is that time progresses at different rates in two frames of reference which move relative to each other. The reason why one cannot be aware of this fact is that relative speeds encountered in everyday life are not comparable to the speed of light.

1.8.7 Length contraction

When a body is at rest relative to an observer, it is easy to measure the distance between points on it. The body is placed next to a ruler and the positions of the two points between which the length is to be measured, are recorded. The results may be checked and rechecked *on condition that no motion exists between the ruler and the body.* The difference between the two positions is, by definition, the required length.

If the body of which the length is to be measured, is in motion relative to the ruler, the procedure will only give the correct results if the two positions are measured *simultaneously in a frame of reference in which the observer and his ruler are at rest.* This is the fundamental fact which governs the transformation of length and the concept of **simultaneity** is inherent to the system in which it is defined. This means that two events which occur simultaneously in one frame of reference, do not necessarily occur simultaneously in another. The reason for this strange fact is that time progresses differently in two inertial systems which are in relative motion.

Consider an object which is at rest along the x'-axis of system S' with its two extremities at positions x'_1 and x'_2 respectively. An observer in S wishes to measure the length of this object relative to his system. In accordance with the foregoing explanation, he has to measure the two positions *simultaneously in his frame of reference.* If the two positions in S' are known (they remain unaltered because the object is at rest relative to S'), the length relative to S may be calculated by means of the Lorentz transformation. Two different relationships between x and x' are available:

$$x = \gamma(x' + ut') \qquad \text{and} \qquad x' = \gamma(x - ut)$$

Both relationships are correct but not equally useful in calculating the end result. The first equation contains t', the time as measured in S' and the observer in S will not be able to use this directly because he needs to make the measurements simultaneously *in his frame of reference.* In order to accomplish this directly, he will have to use the second equation which contains t, the time measured by a clock at rest relative to S.

$$x'_2 = \gamma(x_2 - ut_2) \qquad \text{and} \qquad x'_1 = \gamma(x_1 - ut_1) \qquad 1.8(9)$$

In these two equations x'_1 and x'_2 are the positions of the extremities of the object in system S', x_1 and x_2 their positions in S and t_1 and t_2 the times at which x_1 and x_2 were measured respectively. If the difference between x_1 and x_2 is to be the required length, then it must be chosen that $t_1 = t_2$ (due to the simultaneity requirement). From Equations 1.8(9) we have:

$$x'_2 - x'_1 = \gamma[(x_2 - x_1) - u(t_2 - t_1)] = \gamma(x_2 - x_1) \quad \text{because} \quad t_1 = t_2$$

$$\text{i.e.} \qquad l' = \gamma l \qquad \text{or} \qquad l = l'/\gamma \qquad 1.8(10)$$

Because $\gamma > 1$, it follows that $l < l'$. This means that the observer in S will measure a shorter length *parallel to the x-axis* than that which an observer in S' would measure in his system. The phenomenon is called **length contraction**.

If the observer in S' measures the length of an object which is at rest in S, he will find exactly the same result. The correctness of this statement is left to the reader as an exercise.

Lengths parallel to the other axes are left unaltered by the Lorentz transformation since the relative velocity is parallel to the common x-axis. An ellipse at rest relative to S' with its major axis parallel to the common x-axis will be a circle for an observer in S if the ratio between the major and minor axes and also the relative speed of the two frames of reference allow this. (See problem 54.)

Exercise: Earlier *two* relationships between x and x' were given, and it was explained why the *second one* should be used. The *first* could also have been used to obtain the same result. The reader should try and show how this is possible. It does, of course, involve more work, but it is a worthwhile exercise.

1.8.8 Time dilation

The specification of a position vector and a point in time, defines a unique **event** in that frame of reference. Consider two different events which occur at times t'_1 and t'_2 at the *same position* on the x'-axis in S'. The time interval between the two events is given by $t_2 - t_1$ in S'. We wish to calculate the length of the interval in S. We have two equations which describe time in one system in terms of that in the other. They are as follows:

$$t = \gamma(t' + ux'/c^2) \qquad \text{and} \qquad t' = \gamma(t - ux/c^2)$$

In this case it would be simpler to use the first because it contains x' which remains constant. It may thus be written that:

$$t_2 = \gamma(t'_2 + ux'_2/c^2) \qquad \text{and} \qquad t_1 = \gamma(t'_1 + ux'_1/c^2)$$

in which the symbols have the same meaning as was explained previously. For this case $x'_1 = x'_2$ so that

$$t_2 - t_1 = \gamma(t'_2 - t'_1) \qquad \text{or} \qquad \Delta t = \gamma \Delta t' \qquad 1.8(11)$$

Because $\gamma > 1$, the observer in S will measure a longer interval of time than the one in S'. The practical implication is that an observer in S will state that all clocks in S' run slow. This fact is called **time dilation** and gives rise to interesting results such as the so-called **twin paradox**. The reader is advised to consult more specialised texts[2,3] on this subject.

Exercises:
(1) The length of the beam tube of a particle accelerator is 10 m. Calculate the length of the tube relative to protons which move through it at a constant speed of 0,8c m s^{-1}, in which c represents the speed of light in free space.

Let frame of reference S be at rest relative to the laboratory, and S' at rest relative to the proton beam. The Lorentz constant for the two systems is given by:

$$\gamma = [1 - (0,8c/c)^2]^{-\frac{1}{2}} = \frac{5}{3}$$

In order to calculate the length of the tube in S', the positions of the two extremities have to be measured simultaneously in S'. The transformation which contains t', is the most convenient to use.

$$x_2 = \gamma(x'_2 + ut'_2) \qquad \text{and} \qquad x_1 = \gamma(x'_1 + ut'_1) \qquad \text{with} \quad t'_1 = t'_2$$

From this follows:

$$\begin{aligned} x_2 - x_1 &= \gamma(x'_2 - x'_1) \\ \text{i.e.} \quad 10 &= (5/3)l' \qquad \text{so that} \qquad l' = 6 \text{ m} \end{aligned}$$

According to an observer in S, the time of flight of a proton through the beam tube is given by

$$\Delta t = 10,00/0,8c = 4,2 \times 10^{-8} \text{ s}$$

[2] Young: University Physics (Addison Wesley)
[3] Gamow G: Mr Tompkins in Paperback (Cambridge University Press)

According to an observer moving with the protons, the time of flight through the tube is given by

$$\Delta t' = 6,00/0,8c = 2,5 \times 10^{-8}\ \text{s}$$

(2) A star moving at a speed of 0,6c m s^{-1} away from our solar system, becomes a nova which emits intense light for a period of 100 days as seen by an astronomer on Earth. How long did the emission of the light last relative to the star?

For the two systems $\gamma = [1 - (0,6)^2]^{-\frac{1}{2}} = 1,25$. Choose Earth as S and the star, S'. Because the two events (commencement and end of the emission of light) take place at the same position in S', a transformation which contains x' has to be used.

$$t_2 = \gamma(t'_2 + ux'_2/c^2) \qquad \text{and} \qquad t_1 = \gamma(t'_1 + ux'_1/c^2) \qquad \text{with} \quad x'_1 = x'_2$$

From this follows:

$$\begin{aligned} t_2 - t_1 &= \gamma(t'_2 - t'_1) \\ \text{i.e.}\quad 100 &= 1,25\Delta t' \qquad \text{so that} \qquad \Delta t' = 80\ \text{days} \end{aligned}$$

1.8.9 The Lorentz transformation for velocities

A point moves relative to S at velocity $\bar{v}$ (with components dx/dt, dy/dt and dz/dt). Relative to S', it moves at velocity $\bar{v}'$ (with components dx'/dt', dy'/dt' and dz'/dt'). Calculating the transformation of the components of velocity, is rather more complicated than was the case with the Galilei transformation because both displacements and time intervals differ in the two systems.

$$\begin{aligned}
\text{By definition :}\quad v'_x &= \frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} && \text{(chain rule)} \\
\text{From 1.8(7):}\quad x' &= \gamma(x - ut) && \text{so that} \\
\frac{dx'}{dt} &= \frac{d}{dt}[\gamma(x - ut)] = \gamma(\frac{dx}{dt} - u) = \gamma(v_x - u) && \\
\text{Also from 1.8(7):}\quad t' &= \gamma(t - ux/c^2) && \text{so that} \\
\frac{dt'}{dt} &= \frac{d}{dt}[\gamma(t - ux/c^2)] = \gamma(1 - uv_x/c^2) && \\
\text{therefore}\quad \frac{dt}{dt'} &= \left(\frac{dt'}{dt}\right)^{-1} = [\gamma(1 - uv_x/c^2)]^{-1} && 1.8(12) \\
\text{so that}\quad v'_x &= \frac{\gamma(v_x - u)}{\gamma(1 - uv_x/c^2)} = \frac{v_x - u}{1 - uv_x/c^2} && 1.8(13)
\end{aligned}$$

$$\text{Similarly} \quad v'_y = \frac{dy'}{dt'} = \frac{dy'}{dt}\frac{dt}{dt'} \qquad \text{(chain rule)}$$

$$\text{From 1.8(7):} \quad y' = y \quad \text{so that} \quad \frac{dy'}{dt} = \frac{dy}{dt} = v_y$$

$$\text{From this and 1.8(12):} \quad v'_y = \frac{v_y}{\gamma(1 - uv_x/c^2)} \qquad 1.8(14)$$

$$\text{Similarly:} \quad v'_z = \frac{v_z}{\gamma(1 - uv_x/c^2)} \qquad 1.8(15)$$

The inverse transformation may be written down directly by interchanging the primed and non-primed quantities, and by substituting $-u$ for u. The complete velocity transformation is as follows:

$$\begin{aligned} v'_x &= \frac{v_x - u}{1 - uv_x/c^2} & \qquad & & v_x &= \frac{v'_x + u}{1 + uv'_x/c^2} \\ v'_y &= \frac{v_y}{\gamma(1 - uv_x/c^2)} & & \text{with inverse transformation} & v_y &= \frac{v'_y}{\gamma(1 + uv'_x/c^2)} \\ v'_z &= \frac{v_z}{\gamma(1 - uv_x/c^2)} & & & v_z &= \frac{v'_z}{\gamma(1 + uv'_x/c^2)} \end{aligned} \qquad 1.8(16)$$

1.8.10 The summation rule for velocities

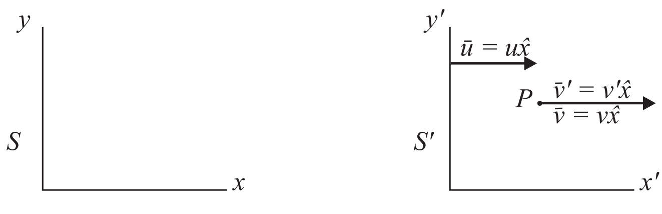


Figure 1.8-6

Consider once again frames of reference S and S', the velocity of S' relative to S, being $\bar{u} = u\hat{x}$. A point P is in motion with velocity $\bar{v}' = v'\hat{x}$ relative to S'. Its velocity relative to S is $\bar{v}$. Since $\bar{v}'$ has only an x-component, only the transformation for this component needs to be applied. From 1.8(16) follows (the equation in the upper line on the right is being applied):

$$v = \frac{v' + u}{1 + uv'/c^2} \qquad 1.8(17)$$

which some prefer to call the **summation rule for velocities**. In fact, it is only a very special application of the Lorentz transformation for velocities. It is enlightening to compare this with the corresponding equation in the Galilei transformation:

$$v = v' + u \qquad 1.8(18)$$

The reader can easily show that 1.8(17) reverts to 1.8(18) when $u \ll c$.

Suppose $v' = c$ as would be the case for a photon (a light particle) emitted from a source at rest at O', the origin of S'. From 1.8(17) the observed velocity in system S may be calculated as follows:

$$v = \frac{c + u}{1 + uc/c^2} = c$$

This result shows that the Lorentz transformation is consistent with its fundamental postulate: The speed of light is Lorentz invariant.

Example: A proton which moves at velocity $0,8c\hat{x}$ relative to the laboratory, collides with a thin target and a K-meson is created. The K-meson moves at velocity $0,9c\hat{x}$ relative to the incident proton beam. Calculate the speed of the meson relative to the laboratory system, i.e. the speed that would be recorded by an instrument at rest in the laboratory.

From 1.8(17)

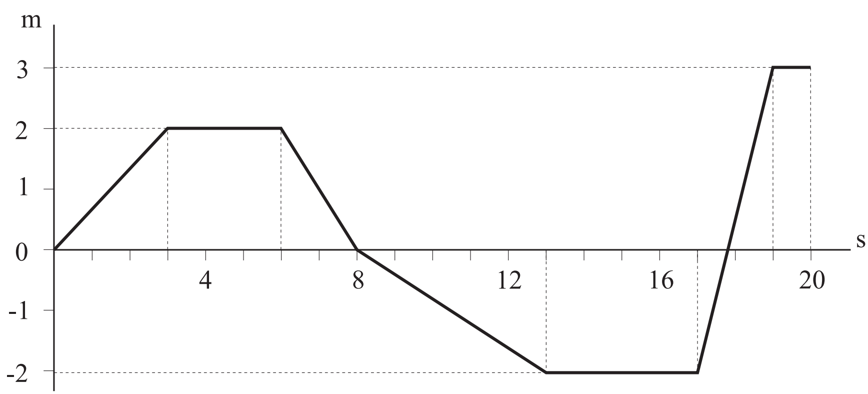
$$\begin{aligned} v &= \frac{0,9c + 0,8c}{1 + (0,9c)(0,8c)/c^2} \\ &= \frac{1,70c}{1,72} = 0,988c \end{aligned}$$

1.9 PROBLEMS: CHAPTER 1

Revision of school physics

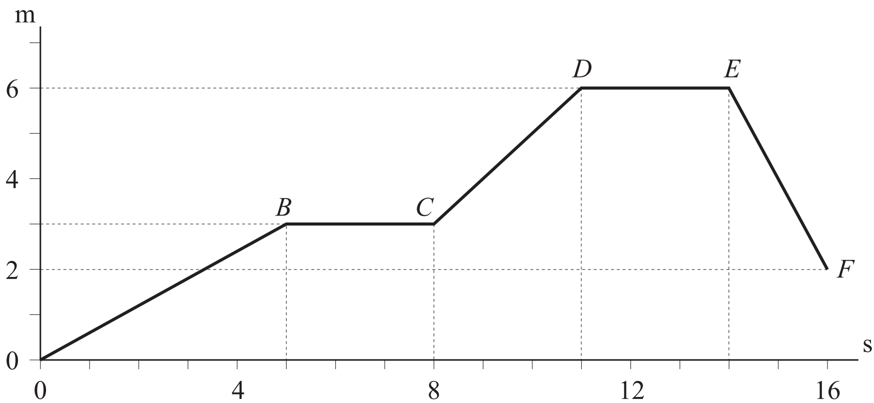
1. An aircraft experiences four successive displacements as follows: $\bar{A} = 200$ km due east, $\bar{B} = 500$ km in direction E 53°N, $\bar{C} = 300$ km due north and $\bar{D} = 900$ km in direction W 30°S. The flight lasts 9,5 h. Choose a frame of reference with origin at the starting point of the journey, $\hat{x}$ east and $\hat{y}$ north. Calculate (a) each displacement in terms of the unit vectors $\hat{x}$ and $\hat{y}$, (b) the position of the aircraft after each of the displacements, (c) the magnitude and direction of the resulting displacement at the end of the journey, (d) the displacement required to return the aircraft to the starting point, (e) the total distance travelled, (f) the average speed for the entire journey, (g) the average velocity for the entire journey, (h) the average velocity during displacement $\bar{B}$ if it is assumed that the speed remains constant throughout the journey.

2.



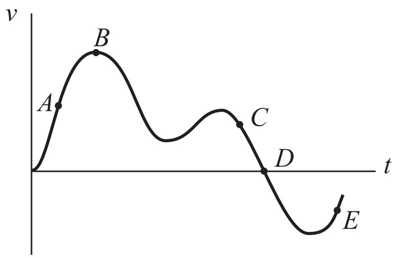
A particle moves along a straight line which has a north-south direction. At instant $t = 0$, the particle is at point P on this line. The above graph gives the position measured from P in metres as a function of time in seconds. Positions north of P are taken as positive. (a) What distance does the point travel during its journey? (b) What is the resultant displacement at the end of the journey? (c) What is the average speed for the entire journey? (d) What is the average velocity over the entire journey? (e) Draw a velocity-time graph for the journey. (f) Draw a speed-time graph for the journey.

3. An object moves along a straight line which has a north-south direction. It starts from the origin, O and consecutively passes through points B, C, D, E and F. In the graph the abscissa gives the time in seconds and the ordinate the position in metres relative to O. All the points indicated on the graph, lie on the same straight line and F is north of O. (a) What is the velocity of the



object between points O and B? (b) What is its velocity between B and D? (c) What is its velocity between E and F? (d) What is the acceleration between C and D? (e) What is the average speed for the entire journey? (f) What is the average velocity for the entire journey? (g) In which cases is one and the same position indicated by different letters? (h) Sketch a velocity-time graph for the entire journey.

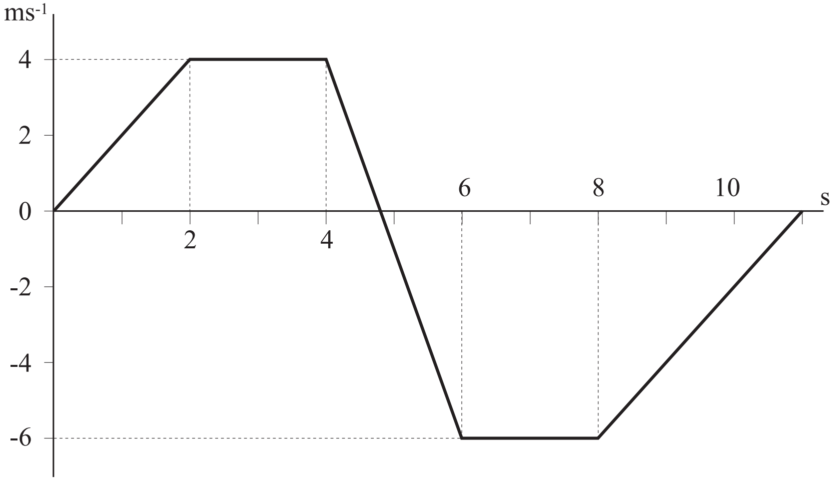
4.



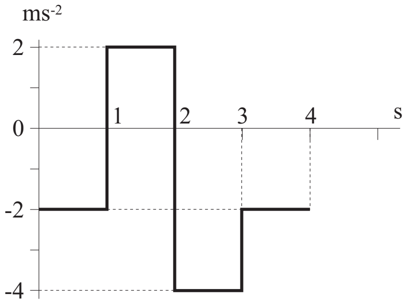
A body moves along a straight line which lies north-south. Quantities measured in a northerly direction are taken as positive and those in the opposite direction, negative. The graph gives the velocity as a function of time. Use the following symbols: N indicates north, S, south and 0 indicates that the appropriate quantity is zero and that the specification of direction is of no consequence.

Complete a table with two columns in which the directions of the velocity and acceleration are indicated by means of the prescribed symbols for A, B, C, D, and E on the graph.

5. The motion of a point is limited to a straight line which lies north-south. Quantities measured in a northerly direction are taken as positive. The graph gives the velocity of the point as a function of time. (a) What is the distance travelled during the entire journey? (b) What is the average velocity during the entire journey? (c) What is the acceleration between t = 2s and t = 4 s? (d) What is the acceleration during the first two seconds? (e) What is the acceleration between t = 4 s and t = 6 s? (f) Draw a graph of the acceleration as a function of time for the entire journey.



6. A particle which moves along a straight line, starts at the origin at instant $t = 0$ with an initial velocity of $+3\ \mathrm{m\,s^{-1}}$. The graph gives the acceleration as a function of time. Calculate the velocity and position at the instants when $t = 1, 2, 3$ and 4 s. Try to obtain the answers by more than one method. (Some answers may be calculated and also be determined by a graphical method.) This is actually four different problems. In each case the final values for a given interval of time, are the initial values for the following interval.



Formal kinematics

7. The motion of a body is limited to the x-axis of a Cartesian co-ordinate system. Its position relative to the origin is given by the vector function $\bar{r} = (30t - 5t^2 - 25)\hat{x}$ m in which the time, t, is measured in seconds. (a) What is the initial position of the body? (b) Calculate the displacement between $t = 2$ s and $t = 5$ s. (c) When does the body pass through the origin? (d) Calculate the velocity of the body as a function of time. (e) What is the initial velocity of the body? (f) When is the body at rest? (g) Calculate the acceleration of the body.

8. The position of a point is given relative to a Cartesian frame of reference by:

$$\bar{r} = (3t^2 - 2t + 3)\hat{x} + (1,5t^2 - 2t + 2)\hat{y} + (-t^2 - t + 1)\hat{z}\ \mathrm{m}$$

in which the time, t, is measured in seconds. (a) Calculate the initial position.

(b) At what distance is the point from the origin at $t = 0$? (c) Calculate the initial velocity. (d) Calculate the initial speed. (e) Calculate the acceleration at time $t = 0$. (f) Calculate the magnitude of the acceleration at time $t = 0$. (f) Where is the point at $t = 1$ s?

9. A particle moves along a space curve with parametric equations:

$$x = 2t^2, \qquad y = t^2 - 4t, \qquad z = 3t - 5$$

in which x, y and z are measured in metres and the time, t, in seconds. (a) Write the position vector in terms of the base vectors. (b) Calculate the initial position of the particle. (c) Calculate the velocity of the particle. (d) Calculate its initial velocity. (e) Calculate its initial speed. (f) Calculate the acceleration. (g) Calculate the component of the velocity in the direction of the vector $\bar{B} = \hat{x} - \hat{y} + 2\hat{z}$, at time $t = 0$ s.

10. The position vector of a point is given by the following function of time:

$$\bar{r} = (2 \sin 4\pi t)\hat{x} + (2 \cos 4\pi t)\hat{y} + (3t)\hat{z} \text{ m}$$

in which time, t, is measured in seconds. (a) Describe the space curve along which it moves. (b) Calculate its velocity. (c) Calculate the speed at $t = 0$. (d). Calculate the magnitude of the acceleration.

11. The velocity of a body is given by $\bar{v} = (-10 - 4t)\hat{x}$ m s^{-1} in which time, t, is measured in seconds. The initial position is $\bar{r}(0) = -12\hat{x}$ m. (a) Calculate the initial velocity of the body. (b) When is the body at rest? (c) Calculate the acceleration. (d) Calculate the position vector of the body. (e) If its acceleration had been the same at two seconds before $t = 0$, where would the body have been then? (f) What would the velocity have been two seconds before the commencement of the measurement of time?

12. The velocity of the centre of mass of a body is given by:

$$\bar{v} = (-6 \sin 2t)\hat{x} + (6 \cos 2t)\hat{y} \text{ m s}^{-1}$$

in which time, t, is measured in seconds. Calculate (a) the speed of the body, (b) the position of the body if it is known that $\bar{r}(0) = 3\hat{x}$ m, (c) the acceleration of the centre of mass, (d) the Cartesian equation of the space curve in the form $y = y(x)$. (e) Show that the position and velocity vectors are orthogonal.

13. A body moves relative to a Cartesian frame of reference. In this system its velocity is given by the following vector function of time:

$$\bar{v} = (-2t - 1)\hat{x} + (6t - 2)\hat{y} + (3t - 2)\hat{z} \text{ m s}^{-1}$$

in which time, t, is measured in seconds. The initial position of the body is $\bar{r}(0) = \hat{x} + 3\hat{y} - 2\hat{z}$ m. (a) Calculate the acceleration. (b) Calculate the position vector as a function of time. (c) When does the body cross the plane $z = 0$? (d) Calculate the angle between the velocity and the acceleration at $t = 0$.

14. The motion of a body is limited to the z-axis of a Cartesian frame of reference. The initial velocity and position are given by $\bar{v}(0) = -3\hat{z}$ m s^{-1} and $\bar{r}(0) = -10\hat{z}$ respectively. The acceleration is time-independent and is given by $\bar{a} = 2\hat{z}$ m s^{-2}. (a) Calculate the velocity of the body as a function of time. (b) When is the body at rest? (c) When is the velocity equal to $5\hat{z}$ m s^{-1}? (d) When is the velocity equal to $-5\hat{z}$ m s^{-1}? (e) When is the speed equal to 7 m s^{-1}? (f) When is the speed equal to -7 m s^{-1}? (g) Calculate the position vector as a function of time. (h) When does the body pass through the origin? (i) What are the displacements during the following time intervals: *(i)* $t = 0$ and $t =$ 3s, *(ii)* $t = 0$ and $t = 4$ s, *(iii)* $t = 1$ s and $t = 4$ s? (j) Calculate directly the velocity as a function of z, not making use of time. (Hint: $a_z dz = v_z dv_z$).

15. The acceleration of a body, is given by $\bar{a} = 4\hat{x} + 7\hat{y} - 4\hat{z}$ m s^{-2}. Its initial velocity is $\bar{v}(0) = -3\hat{x} + 6\hat{y} + 2\hat{z}$ m s^{-1} and its initial position, $\bar{r}(0) = 2\hat{x} - 2\hat{y} + \hat{z}$ m. (a) Calculate the magnitude of the acceleration. (b) Calculate the velocity as a function of time. (c) Calculate the initial speed. (d) Calculate the angle between the velocity and acceleration vectors at $t = 0$ s. (e) Calculate the position vector as a function of time. (f) Calculate the displacement in the time interval $t = 0$ and $t = 2$ s. (g) Calculate the angle between the acceleration and position vectors at time $t = 0$ s.

16. A body is projected vertically upwards with initial velocity $\bar{v}(0) = 50\hat{z}$ m s^{-1}. Gravitational acceleration is $\bar{g} = -10\hat{z}$ m s^{-2}. Choose the projection point on the ground as origin and $t = 0$ when the body leaves this position. (a) Calculate the velocity as a function of time. (b) Calculate the position as a function of time. (c) When does the body reach maximum height? (d) When is the body 80 m above ground level? (e) When does the body have a velocity of $20\hat{z}$ m s^{-1}? (f) When does the body have a velocity of $-20\hat{z}$ m s^{-1}? (g) At what velocity does the body hit the ground? (h) If the body falls into a mine shaft rather than striking the ground, when will it reach a depth of 120 m? (i) Calculate the velocity as a function of z, without using time.

17. The acceleration of a body is $\bar{a} = -10\hat{y}$ m s^{-2}. At $t = 20$ s, its velocity is given by $\bar{v}(20) = -120\hat{y}$ m s^{-1} and its position by $\bar{r}(20) = -400\hat{y}$ m. (a) Calculate $y = y(t)$. (b) When is y a maximum? (c) What is the maximum value of y? (d) Calculate the velocity as a function of time. (e) Calculate the initial velocity. (f) Calculate the velocity when y is a maximum.

18. A balloon rises vertically at a constant speed of 4 m s^{-1} when a sand bag which is used as ballast is dropped from the passenger basket from a height of 80 m above ground level. Disregard frictional drag and calculate the speed at which the bag strikes the ground. $g = 10$ m s^{-2}.

19. A person runs as fast as possible to catch a bus. When he is 24,3 m from the bus, it departs at a constant acceleration of 0,6 $\mathrm{m\,s^{-2}}$. Calculate the minimum speed of the person (assume it to be constant) which will just enable him to touch the bus. Try to solve the problem in two different ways: *(i)* The velocity of the man is $\bar{v} = v\hat{x}$ and v is unknown. Choose a frame of reference on the ground at the position where the man was when the bus departed. *(ii)* Repeat the problem with the frame of reference on the bus and at rest relative to it. It is possible to make the calculations in this frame of reference without the use of time.

Ballistics or Projectile motion

20. Describe the motion of a projectile over a horizontal plane with origin at the launching point, $\hat{x}$ horizontally forwards and $\hat{y}$ vertically upwards. Gravitational acceleration is $\bar{g} = -10\hat{y}$ $\mathrm{m\,s^{-2}}$. The initial speed is v_0 $\mathrm{m\,s^{-1}}$ and the elevation θ. Choose $t = 0$ when the projectile is launched. (a) Write the initial velocity, $\bar{v}(0)$, in terms of the base vectors of the frame of reference. (b) Calculate the velocity as a function of the time. (c) Calculate the position as a function of the time. (d) Calculate the Cartesian equation of the space curve in the form $y = y(x)$. (e) Calculate the range of the projectile over a horizontal field and also the value of the elevation for which it will be a maximum. Calculate from first principles, i.e. do not use formulas which you might remember. (f) Calculate the maximum height reached over the horizontal field. (g) Calculate the angle of inclination as a function of time, t, and also as a function of the horizontal component of the position vector, x. Use these results to determine when and for what value of x, the projectile will reach its maximum height.

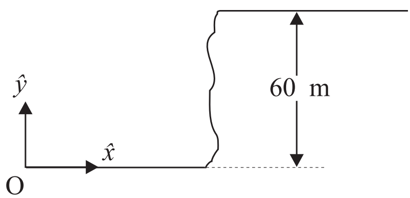
21. A projectile is launched at 50 $\mathrm{m\,s^{-1}}$ and an elevation of 40°. $g = 10$ $\mathrm{m\,s^{-2}}$ Choose a Cartesian frame of reference with origin at the launching point, $\hat{x}$ horizontally forwards and $\hat{y}$ vertically upwards. (a) Write down the acceleration vector in terms of the base vectors. (b) Write down the initial velocity in terms of the base vectors. (c) Calculate the velocity as a function of time. Choose $t = 0$ when the projectile leaves the origin. (d) Calculate the position vector as a function of time. (e) Calculate the y-component of the velocity, v_y, as a function of the height, y. (f) Calculate the maximum height of the projectile over a horizontal field which contains the launching position. (g) Calculate the Cartesian equation of the space curve in the form $y = y(x)$. (h) Calculate the range over a horizontal field which contains the launching position. (i) Calculate the time of flight over this horizontal field.

22. A projectile is launched over a horizontal field at 40 $\mathrm{m\,s^{-1}}$ and an elevation of 60°. Choose any convenient frame of reference and calculate the equation of its space curve. $g = 10$ $\mathrm{m\,s^{-2}}$. Calculate in as many ways as are possible the:

(a) maximum height, (b) time of flight, (c) range.

23. A projectile is launched over a horizontal field from the top of a 60 m high tower. The initial speed is 40 $\mathrm{m\,s^{-1}}$ and the elevation, 30°. $g = 10\ \mathrm{m\,s^{-2}}$. Choose a Cartesian frame of reference with $\hat{x}$ horizontally forwards and $\hat{y}$ vertically upwards. Choose $\bar{r}(0) = \bar{0}$. (a) Calculate the velocity as a function of time. (b) Calculate the position as a function of time. (c) Calculate the equation of its flight path in the form $y = y(x)$. (d) Use the flight-path equation to calculate the range over the horizontal field on which the tower stands. (e) Calculate the azimuth angle at which the projectile strikes the ground.

24.

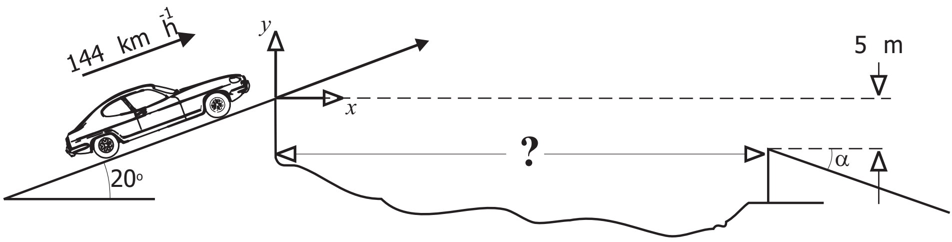


A projectile is launched at time $t = 0$ from the horizontal plane below a precipice which is 60 m deep. The launching point is far enough from the edge of the escarpment for the projectile to miss it, but still reach a target on the plateau above. The initial speed of the projectile is 56,57 $\mathrm{m\,s^{-1}}$ and the elevation angle, 45°. Choose a Cartesian frame of reference with origin at the launching point, $\hat{x}$ horizontally forwards and $\hat{y}$ vertically upwards. $g = 10\ \mathrm{m\,s^{-2}}$. Frictional drag may be disregarded.
(a) Calculate the velocity as a function of time. (b) Calculate the position as a function of time. (c) Calculate the time of flight of the projectile. (d) Calculate the maximum height that the projectile will reach above the plateau. (e) Calculate the range. (f) Assume that the precipice is vertical. Calculate the shortest distance between the launching point and the base of the precipice that will allow the projectile just to miss the edge of the escarpment and still reach the plateau. The initial conditions remain the same.

25. An aircraft flies horizontally at 100 $\mathrm{m\,s^{-1}}$ at height 2000 m above ground level when it releases a bomb. $g = 10\ \mathrm{m\,s^{-2}}$. Disregard frictional drag and calculate the time of flight of the bomb. At what speed does it strike the ground? What is the range of the bomb?

26. A mortar with muzzle speed 300 $\mathrm{m\,s^{-1}}$ is to strike a target at 7794 m over a horizontal field. Calculate the possible elevation angles which may be used. Also calculate the maximum height in each case.

27. You are consultant to the famous dare-devil driver Mr Evel Knievel who has received an offer from a motor car manufacturer and you have to determine if the task can be successfully executed. As an advertisement gimmick Mr



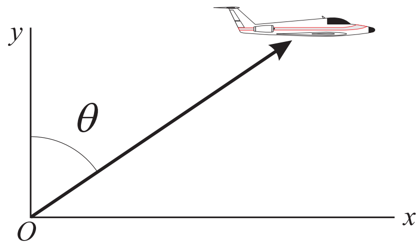
Knievel is required to perform a jump in a new model car across a tributary of the Fish River canyon in Namibia which is about 100 m wide. Mr Knievel believes that he will be able to do it by using a ramp with an incline of 20° at an initial speed of 144 $\mathrm{km\,h^{-1}}$. The nearest point of the landing ramp is 5 m below the launching point as is shown in the sketch. The motor car will be fitted with aerofoils to keep its body parallel to its flight path. As *a first approximation*, you use the following *model*: Choose a Cartesian frame of reference with origin at the launching point, $\hat{x}$ horizontally forwards and $\hat{y}$ vertically upwards. $g = 10\ \mathrm{m\,s^{-2}}$. Write down the initial velocity, initial position and acceleration vector in the chosen system. Disregard frictional drag. (a) Calculate the velocity and position as functions of time in the chosen system. (b) Calculate the flight-path in the form $y = y(x)$. (c) Use the flight path equation to calculate the inclination angle of the path as a function of x. (d) Determine if it will be possible to erect a landing ramp on the other side of the tributary of which the beginning is 5 m lower than the launching point. The width of the abyss is not more than 100 m. If it seems to be possible, calculate where the beginning of the landing ramp should be. (e) If the feat is deemed possible, what would your advice be as to the choice of the angle of the landing ramp (α in the sketch.)

Challenge problems

28. At $t = 0$ s a point begins from the positive x-axis of a Cartesian system and moves on a circle with radius 12 m of which the centre is at the origin. It moves in a positive (anticlockwise) sense and the arc length is given by $s = t^3 + 3t$ m in which the time, t, is measured in seconds. Calculate the velocity of the point at $t = 2$ s.

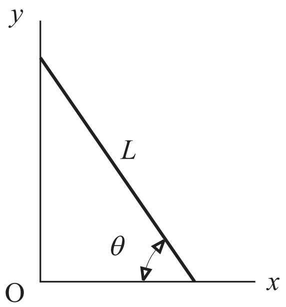
29. A particle describes a space curve with equation $y = (3/4)x - (5/64)x^2$ in which x and y are measured in metres. The x-component of the velocity is constant and equal to $+8\ \mathrm{m\,s^{-1}}$. It is known that $\bar{r}(0) = \bar{0}$. Calculate $\bar{r}$, $\bar{v}$ and $\bar{a}$.

30.



An aircraft flies horizontally along a straight line at a speed of 120 $\mathrm{m\,s^{-1}}$ at a height of 100 m above ground level. Calculate the angular velocity, $d\theta/dt$, of the position vector relative to a stationary observer on the ground. Calculate it as a function of θ and its value when $\theta = 60°$. (Hint: Use the rule: $(d/dt)(\tan\theta) = [(d/d\theta)(\tan\theta)] \times (d\theta/dt)$.

31.



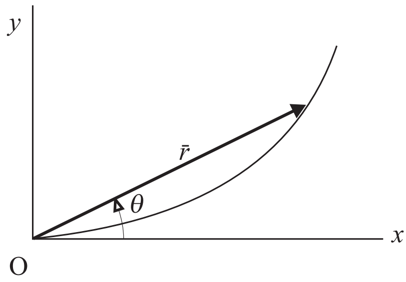
A ladder AB is L metres long and stands against a vertical wall. Choose a Cartesian frame of reference with origin where the wall meets the floor with $\hat{x}$ horizontally away from the wall and $\hat{y}$ vertically upwards as shown in the sketch. The foot of the ladder starts against the wall at $t = 0$ and slides away at a constant speed of $0,5\ \mathrm{m\,s^{-1}}$. The angle between the floor and the ladder is represented by θ. Calculate the velocity of the top of the ladder as a function of time. Calculate the velocity when $\theta = 70°$.

Also calculate the acceleration of the top of the ladder as a function of time. (Hint: Write down the position of the foot of the ladder as a function of time. Then calculate the position of the top by means of the theorem of Pythagoras.)

32.The units along the axes of a Cartesian frame of reference, are metres. A particle describes the path $y = 4x^2$ in such a way that the x-component of the velocity is constant and equal to 2 $\mathrm{m\,s^{-1}}$. Calculate (a) the position vector as a function of time. Take $\bar{r}(0) = \bar{0}$, (b) the component of the acceleration parallel to the velocity vector (i.e. the tangential acceleration) when x = 0,5 m, (c) the component of the acceleration perpendicular to the space curve (i.e. centripetal acceleration) when x = 0,5 m. (Hint: First calculate $x = x(t)$ and then $y = y(t)$. After that $\bar{r} = \bar{r}(t)$ is known and the rest may be calculated.)

33. A particle moves along the x-axis of a Cartesian frame of reference with acceleration given by $\bar{a} = 6x^{\frac{1}{3}}\hat{x}\ \mathrm{m\,s^{-2}}$ in which x is measured in metres, When $t = 2$ s, its position is $\bar{r}(2) = 27\hat{x}$ m and its velocity, $\bar{v}(2) = 27\hat{x}\ \mathrm{m\,s^{-1}}$. Calculate the position, velocity and acceleration of the particle at $t = 4$ s. (Hint: Use the differential equation $a_x\,dx = v_x\,dv_x$ and integrate. Furthermore: $v_x = dx/xt$.)

34.



A particle moves in the $z = 0$ plane along a curve of which the equation in plane polar co-ordinates, is $r = 2\theta$. In this equation, r, is the length of the position vector (or *radius vector* as it is sometimes called) and θ the azimuth angle. It is known that $\theta = t^2$ rad, in which the time, t, is measured in seconds. Calculate the velocity of the particle when $\theta = 60°$. Also calculate its speed and direction of motion at this position.

(Hint: First write down $r = r(t)$, then calculate $x = x(t)$ and $y = y(t)$. Then $\bar{r} = \bar{r}(t)$ is known.)

35. A particle moves along the y-axis of a Cartesian frame of reference with acceleration $\bar{a} = 2v^{\frac{1}{2}}\hat{y}$ m s^{-2}, in which its speed, v, is measured in m s^{-1}. When $t = 2$ s, the velocity is $\bar{v}(2) = 16\hat{y}$ m s^{-1}, and when $t = 1$ s, its position is $\bar{r}(1) = 9\hat{y}$ m. Calculate the position, velocity and acceleration of the particle when $t = 3$ s.

36. A particle is limited to the x-axis of a Cartesian frame of reference. The velocity component along this axis is always numerically equal to the value of x, i.e. $\bar{v} = v\hat{x} = x\hat{x}$. Calculate the acceleration of the particle when $x = 5$ m. How long does it take the particle to move from $x = 3$ m to $x = 5$ m?

Relativity

37. A boy, sitting in a bus which is moving at 20 m s^{-1} along a straight line, folds a paper aeroplane which he projects at 2 m s^{-1}. Calculate the speed of the plane relative to the ground if the direction in which it is projected, is (a) parallel to, (b) antiparallel to, (c) perpendicular to the direction in which the bus is moving.

38. At a given instant, ship A is 8 sea miles due east of ship B. A is moving at 20 knots west and B at 15 knots in direction N 30°E. Calculate the velocity of B relative to A. Calculate the shortest distance that will separate the two ships. How long will it take for this relative position to be reached?

39. An aircraft flies at 200 knots due south relative to the air. A wind is blowing at 40 knots west. Calculate the velocity and speed of the craft relative to the ground.

40. The velocity of the wind is 25 km h^{-1} in direction 120° (0° or 360° is north and 90° east according to the notation used by navigators). The air speed of a light aircraft is 145 km h^{-1}. Calculate the bearing (compass reading) in order that the aircraft may fly due north relative to the ground. Calculate its ground speed when it is flying in this way.

41. An official employed by a mine at Hotazel in the North-west Cape wishes to visit his farm in the Piet Retief district, using his private aircraft. Piet Retief is due east from Hotazel at a distance of 1050 km along the geodesic through these two places. In order to draw up a flight plan, the pilot requests information from the weather bureau. According to the bureau a wind of 20 km h^{-1} in direction E 25°S prevails along the entire route. Calculate the compass bearing for the flight, the ground speed and time of flight for the journey. The air speed of the craft is 240 km h^{-1}.

42. The axes of Cartesian frame of reference, S (origin O) are parallel to those of S' (origin O'). When $t = t' = 0$ their origins coincide. The position vector of point P as measured relative to S, is $\bar{r} = 3\hat{x} + 2\hat{y} + 4\hat{z}$ m. Calculate the position vector $\bar{r} = \bar{r}(t)$ of P relative to S'. The velocity of O' relative to S, is (a) $\bar{u} = 7\hat{x}$ m s^{-1}, (b) $\bar{u} = 2\hat{x} - \hat{y} - 2\hat{z}$ m s^{-1}.

43. The axes of Cartesian frames of reference S and S' are parallel and the origins O and O' coincide at $t = t' = 0$. At $t = 0$, point M has as position vector $\bar{r}(0) = 2\hat{x} - \hat{y} + 2\hat{z}$ m and moves at constant velocity $\bar{v} = 6\hat{x} - 2\hat{y} + 3\hat{z}$ m s^{-1} relative to S. (a) Calculate $\bar{r} = \bar{r}(t)$ relative to S. (b) Calculate $\bar{r}' = \bar{r}'(0)$ (c) Calculate $\bar{v}' = \bar{v}'(t')$ and $\bar{r}' = \bar{r}'(t')$ for the cases where the velocity of O' relative to S is given by (i) $\bar{u} = 7\hat{x}$ m s^{-1}, (ii) $\bar{u} = 2\hat{x} - \hat{y} + 2\hat{z}$ m s^{-1}.

44. On the moon the magnitude of gravitational acceleration is 1,8 m s^{-2}. A mooncraft manned by an observer, moves across the lunar surface along a horizontal straight line at 3 m s^{-1} . At $t = 0$, the observer throws an object vertically upwards at an initial speed of 2 m s^{-1}. (a) Calculate the position vector of the object in a frame of reference which is at rest relative to the moon with its origin where the object was thrown. Choose $\hat{x}$ horizontally forwards and $\hat{y}$ vertically upwards. (b) Calculate the position vector of the object in a frame of reference at rest relative to the observer, with axes parallel to those of the system in (a) and of which the origins coincide at $t = 0$. (c) Describe the space curve in each frame of reference.

45. While an angler is rowing his boat upstream, his thermos flask which is empty enough to float on the water, accidentally falls overboard. After he has travelled for 15 minutes, he discovers his loss. He immediately turns the boat around and rows at the same speed relative to the water. He finds the flask 1,5 km downstream from the place where it fell overboard. Calculate the speed of the stream.

46. A particle is at rest relative to a given frame of reference at position $\bar{r} = \hat{x} + 2\hat{y} + 3\hat{z}$ m. The axes of a second system are parallel to those of the first and it moves at relative velocity $\bar{u} = 3\hat{x}$ m s^{-1}. At $t = t' = 0$ the axes of the two frames of reference coincide. Calculate the position vector of the particle relative to the second system at times $t' = 0$, $t' = 2$, $t' = 10$ and $t' = 100$ s, for a (a) Galilei transformation, (b) Lorentz transformation.

47. Repeat the previous question for $\bar{u} = 0,8\,c\hat{x}$ in which $c = 3 \times 10^8$ m s^{-1} = speed of light in free space.

48. A metre stick is at rest relative to frame of reference R' which is moving at velocity $\bar{u} = 0,6\,c\hat{x}$ m s^{-1} relative to R. The two sets of axes are parallel and their origins coincide at $t = t' = 0$. The extremities of the stick are at positions $x'_1 = 7$ m and $x'_2 = 8$ m in R'. Calculate the co-ordinates of these points as functions of time in frame R. Use the (a) Galilei transformation, (b) Lorentz transformation. (c) Which of the two transformations is correct?

49. A rod is at rest relative to a frame of reference P' which is moving at velocity $0,8c\hat{x}$ m s^{-1} relative to frame P. The axes of the systems are parallel and their origins coincide at the beginning of their time scales. Relative to P', the positions of the two extremities are $\bar{r}'_1 = 1,5\hat{x}' + 1,0\hat{y}'$ m and $\bar{r}'_2 = 3,5\hat{x}' + 2,5\hat{y}'$ m. Calculate the position vectors of the two extremities of the rod relative to P at instant $t = 0$ according to a (a) Galilei transformation, (b) Lorentz transformation. (c) At what angle is the rod to the $\hat{x} - \hat{x}'$-direction in each case? (d) Which answers are correct?

50. Observers John and Mandy are at rest in frames of reference R and R' at positions $x = 5$m and $x' = 3$ m respectively. The axes coincide at the common beginning of their time scales. The velocity of R' is $\bar{u} = 0,6\,c\hat{x}$ m s^{-1} relative to R. John strikes a match at time $t_1 = 2$ s, lights a cigarette and extinguishes the flame at $t_2 = 5$ s. What corresponding times does Mandy measure in each case according to a (a) Galilei transformation, (b) Lorentz transformation? (c) For what length of time will the match burn in each case according to Mandy? Which answers are correct?

51. The axes of S and S' coincide at $t = t' = 0$. The velocity of S' is $\bar{u} = 0,6\,c\hat{x}$ m s^{-1} relative to S. Calculate the value of the Lorentz constant, γ, for the two systems. π-mesons are at rest relative to S'. An observer in S' finds that the average lifetime of a π-meson is $2,6 \times 10^{-8}$ s. Calculate the average lifetime of these particles as observed by an observer in S. Give a full explanation of the calculation method.

52. What will the relative speed of two frames of reference be in order that a time interval of Δt measured in one, will be $2\Delta t$ in the other?

53. The length of a moving metre-stick is measured as 0,6 m. Calculate its *minimum* speed relative to the observer. Why do we refer to the *minimum* speed?

54. An ellipsoid has a semi-major axis of 40 mm and a semi-minor axis of 20 mm. It moves parallel to its major axis. Calculate the speed that it should have in order to appear as a sphere to the observer.

55. The beam tube of an accelerator has a length of 100 m. Protons move through it at a speed of $0,8\,c\ \mathrm{m\,s^{-1}}$ relative to the laboratory system. $c = 3\times10^8$ $\mathrm{m\,s^{-1}}$ = speed of light in free space. Calculate the length of the tube relative to the proton beam. How long does it take a proton to move right through the tube according to an observer at rest relative to the laboratory and also according to one moving with the proton beam?

56. The edge of a bank note which coincides with the x'-axis of frame of reference S' relative to which it is at rest, is 135,5 mm and that parallel to the y'-axis, 70 mm. Which lengths are measured in S if the velocity at which the note is moving towards the receiver of revenue is (a) $0,8\,c\hat{x}$, (b) $0,9\,c\hat{x}$, (c) $0,99\,c\hat{x}$? c = speed of light in a vacuum = $3\times10^8\ \mathrm{m\,s^{-1}}$.

57. A spaceship is on its way to a distant star at a speed of $0,8\,c\ \mathrm{m\,s^{-1}}$ relative to Earth. How long does it take to travel a distance of 200 m (as measured by an observer on Earth) according to (a) the space traveller on the ship, (b) the observer on Earth?

58. A proton with velocity $0,7\,c\hat{x}\ \mathrm{m\,s^{-1}}$ relative to the laboratory, strikes a thin target and produces a K-meson which moves at $0,85\,c\hat{x}\ \mathrm{m\,s^{-1}}$ relative to the incident proton beam. What speed will a scientist in the laboratory measure? Show the calculation.

59. According to a physicist, K-mesons are moving at a velocity of $0,75\,c\hat{x}\ \mathrm{m\,s^{-1}}$ relative to the laboratory. The proton beam from which they were created, has a velocity of $0,6\,c\hat{x}\ \mathrm{m\,s^{-1}}$ relative to the laboratory. Calculate the velocity of the mesons relative to the proton beam.

Chapter 2

THE DYNAMICS OF A POINT MASS

2.1 Mass, momentum & Newton's laws

Kinematics which was treated in chapter 1, is that section of physics which deals with the measurement of motion. The positions, velocities and accelerations of points were measured and described. The reasons for motion or changes of motion were irrelevant to these studies. In this section which is known as **dynamics** (Greek: $\delta\upsilon\nu\alpha\mu\iota\varsigma \equiv$ force), the reasons for motion and changes in it are studied.

In this study the concept of **point mass** will be used quite often. This is necessary because bodies which occupy space, have properties which cannot be taken into account at this stage. In nature point masses do not occur. A point mass is an idealised mathematical representation of a body in which its spatial qualities with the exception of position, may be disregarded. The nucleus of an atom which is extremely small and also extremely dense, is very well represented by this concept. The **centre of mass** of a body (definition and treatment in chapter 3) may, under suitable circumstances, be thought of as though all the mass of the body is concentrated there and then the dynamics of a point mass will be applicable to the centre of mass of the body.

When the position vector of a body changes, the body is said to experience **translation**. The word translation is usually used to distinguish a different kind of motion which is called **rotation** and which will be treated in chapter

3. In that chapter it will also be shown that the translation and rotation of a point mass, are two different ways to describe one and the same motion.

From everyday experience it is known that every material object possesses **inertia**, a property which may be described as its "unwillingness" to change its state of motion. This property was formulated by Sir Isaac Newton (1642 - 1727) as one of his famous laws of motion. The inertia of a material object is characterised by a quantity knows as its **mass**. The SI unit of mass is **kilogram (kg)**. The concept of inertia will become clearer once Newton's laws of motion are treated and discussed in more detail.

A most fundamental fact of nature is that a material object can experience a change in its motion only while it is engaged **in an interaction** with another object. Interaction is usually described by means of a mathematical concept which is known as **force**. A body which experiences no interaction with another body, is known as a **free body** and its motion is described as being **force-free**. At this stage it will have to suffice to think of a force as the effort exerted when a person pushes against or pulls on an object by applying his muscles.

Hooke's Law (announced 1676) describes the **elastic deformation** of body when forces are applied to it. It will be dealt with in more detail later in this chapter. An elastic deformation is one that disappears completely when the applied forces which caused it, are removed. This law supplies the possibility to measure force or initially, to duplicate an unknown force. If a given force extends a helical spring by a certain length, double that force will cause double the extension. Similarly, double the force will be required to extend two similar springs **in parallel** under the same conditions.

Once a unit of force is defined, an experiment will be described to illustrate Hooke's Law. For a helical spring this law is as follows: *If a force is applied to a helical spring to extend it within its elastic limit, the extension is directly proportional to the magnitude of the force causing it.* The spring balance which may be used to measure force, makes use of this principle.

2.1.1 Momentum

The **linear momentum** of a body is defined as its mass multiplied by its velocity. This is a most important quantity in dynamics since it contains two properties of a body in motion; its velocity and its opposition to change in velocity (inertia or mass). The symbol $\bar{p}$ is usually used to indicate linear momentum (momentum in short):

$$\bar{p} = m\bar{v} \qquad 2.1(1)$$

in which m represents its mass and $\bar{v}$ its velocity. Since mass is a positive scalar quantity and velocity a vector, momentum will always be a vector parallel to the velocity. We may express this fact by the equation $\hat{p} = \hat{v}$.

The SI units of momentum are $\mathrm{kg\,m\,s^{-1}}$, and $[p] = [MLT^{-1}]$.

Examples: 1. A mass of 12 kg has a velocity of $7\hat{x}$ $\mathrm{m\,s^{-1}}$. Its momentum is $\bar{p} = 12 \times 7\hat{x} = 84\hat{x}$ $\mathrm{kg\,m\,s^{-1}}$. A mass of 4 kg would have the same momentum if its velocity were $21\hat{x}$ $\mathrm{m\,s^{-1}}$.

2. A mass of 3 kg has a velocity of $\bar{v} = -3\hat{x} + 6\hat{y} - 2\hat{z}$ $\mathrm{m\,s^{-1}}$. Calculate (a) its momentum, (b) the magnitude of its momentum.

(a) $\bar{p} = 3 \times (-3\hat{x} + 6\hat{y} - 2\hat{z}) = -9\hat{x} + 18\hat{y} - 6\hat{z}$ $\mathrm{kg\,m\,s^{-1}}$

(b) $p = |\bar{p}| = [9^2 + 18^2 + 6^2]^{\frac{1}{2}} = 21$ $\mathrm{kg\,m\,s^{-1}}$.

2.1.2 Newton's first and third laws

Translated from the original Latin, in which Newton's *Philosophiae Naturalis Principia Mathematica* (1686), (known as *Principia* in short) was written, the **first law** of motion is as follows: **Each body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed on it.**

The content of this important law is not at all self-evident. It requires some qualifications. Before that is done, the **third law** is stated: **To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.** This is a direct translation from Latin. In modern English, we may explain it as follows: Single forces do not occur in nature; they always exist in pairs which are equal in magnitude but opposite in direction. They also conform to the following: (i) The two opposite forces act along one and the same straight line. (ii) The two forces *always* act upon two different bodies. (iii) The two forces originate from the same interaction. It will later be shown that this law leads to the principle of conservation of linear momentum.

What Newton describes in his first law as *uniform motion in a straight line* is exactly what is meant by *motion at a constant velocity*.

The next concept which needs to be clarified, is that of *body*. If it is equated to the concept of *point mass*, Newton's first law simply means that **inertial systems** exist in which, by definition, force-free motion must also be acceleration-free. (See subsection 1.8.1) From the study of relativity, is should be clear to the reader that if a body does not accelerate in one inertial frame of reference, it also does not accelerate in any other inertial frame. It also implies that for each such body an inertial frame of reference can be found in which it will be at rest.

The word *body* can be taken to mean any collection of material that one wishes to call a body. A stoppered flask filled with helium gas may be described as a body. On the other hand, it will be perfectly in order to describe only the stopper of the bottle as a body, also one of the helium atoms or even a nucleus of one of the atoms.

If Newton's first law is to be applied to a body which is thus defined, some forces which act on it will have to be excluded because they will not be able to alter its state of motion: (i) Forces which are cancelled by other forces. (ii) Forces which one part of the body exerts on another part of the same body.

Forces of which the effects are cancelled by other forces, cannot influence the motion of a body. Although Newton did not state it in as many words when he mentioned forces in his first law, he actually meant **unbalanced forces**.

The forces which one part of a body exerts on other parts, are called **internal forces**. The helium atoms in the flask mentioned previously, are constantly in motion and collide with each other and also with the flask. As a result of these collisions the individual atoms experience large changes in velocity. The flask with its contents, however, remains in its state of motion or rest if no unbalanced external forces act on it. What Newton also did not state in as many words, is that the forces to which he referred, are **external forces**. Because the entire flask with its contents was chosen to be a body, the forces exerted by the individual atoms on each other and on the flask, are internal forces which cannot change its state of motion.

The momentum of a body is a quantity that supplies more information about the behaviour of a body than its velocity alone. It is common knowledge that it is more difficult to change the motion of a body with large mass than one which has less. Because the only difference between momentum and velocity is a constant factor (the mass), it is correct to state that it follows from the discussion of Newton's first and third laws that *the momentum of a body will remain constant unless acted on by an unbalanced external force.* It is equivalent to the statement that the momentum of a **closed system** remains unaltered. In this form it is known as the **principle of momentum conservation.** This principle will be discussed in greater detail in the section on collisions. The following example illustrates the use of this principle:

Example:

Two skaters, A (mass 64 kg) and B (mass 72 kg) stand facing each other. They are in rest relative to the ice rink.B pushes A and they move away from each other. A moves backwards at 1,8 m s^{-1}. Calculate the velocity of B after the interaction. Disregard friction.

Choose a frame of reference with $\hat{x}$ in the direction of A's motion after the interaction. Firstly consider A and B to be parts of one body. When A exerts a force on B, the latter will exert a force of equal magnitude but in the opposite direction on A (action and reaction). According to the choice, these forces are *internal* forces which cannot change the momentum of the body. Before the interaction the body possessed no momentum because both A and B were at rest and the absence of unbalanced external forces will compel it to remain in that state. From the principle of momentum conservation it follows:

Let the velocity of B after the interaction be $\bar{v}$.

$$\begin{aligned} \text{momentum after interaction} &= \text{momentum before interaction} \\ 64 \times 1,8\hat{x} + 72\bar{v} &= \bar{0} \\ \text{from which it follows that} \qquad \bar{v} &= -1,6\hat{x} \quad \text{m s}^{-1} \end{aligned}$$

The problem can be treated differently by choosing that A and B are different bodies. The force which A experiences during the interaction, is then an *unbalanced external force* which changes its momentum.

$$\begin{aligned} \text{change in } A\text{'s momentum} &= A\text{'s final momentum} - A\text{'s initial momentum} \\ \Delta\bar{p} &= 64 \times 1,8\hat{x} - \bar{0} \\ &= 115,2\hat{x} \quad \text{kg m s}^{-1} \end{aligned}$$

B also experienced an external force and a change in momentum.

$$\begin{aligned} \text{change in } B\text{'s momentum} &= B\text{'s final momentum} - B\text{'s initial momentum} \\ \Delta\bar{p} &= 72 \times (-1,6\hat{x}) - \bar{0} \\ &= -115,2\hat{x} \quad \text{kg m s}^{-1} \end{aligned}$$

This highlights a different aspect of the principle: When two bodies interact, the changes of their momenta are equal in magnitude but opposite in sign.

2.1.3 Newton's second law

Newton's second law is a statement about what happens when a body experiences an unbalanced external force. In a certain sense it may be thought of as a definition of force which follows from the principle of momentum conservation.

It may be formulated as follows: **When an unbalanced external force acts on a body, its momentum changes. The time rate at which its momentum changes, is directly proportional to the force.** If the force is represented by $\bar{F}$, this law may written in the following way:

$$\bar{F} \propto \frac{d\bar{p}}{dt} \qquad 2.1(2)$$

This proportionality can be converted to an equality by the introduction of a proportionality constant as follows:

$$\bar{F} = k\frac{d\bar{p}}{dt} = k\frac{d}{dt}(m\bar{v}) = k(\frac{dm}{dt}\bar{v} + m\frac{d\bar{v}}{dt}) \qquad 2.1(3)$$

If the mass of the body remains constant, then $dm/dt = 0$. For this special case equation 2.1(3) reduces to the following:

$$\bar{F} = km\frac{d\bar{v}}{dt} = km\bar{a} \qquad 2.1(4)$$

Up to this stage a unit for force has not been defined and equation 2.1(4) provides the opportunity to do just that. The proportionality constant may be chosen as one and the unit of force may be defined as follows: A force of one **newton (N)** is the force which will cause an acceleration of 1 $\mathrm{m\,s^{-2}}$ if the mass of the body on which it acts, is 1 kg.

Thus 1 newton = 1kilogram × 1 metre second^{-2}

The unit *newton* may be thought of as an abbreviation for $\mathrm{kg\,m\,s^{-2}}$. The dimensions of force are as follows: $[F] = [M\,L\,T^{-2}]$.

Examples:

1. The velocity of a body with mass 12 kg, changes from $6\hat{x}$ $\mathrm{m\,s^{-1}}$ to $4\hat{x}$ $\mathrm{m\,s^{-1}}$ in 4 s. It may be assumed that the change in velocity is caused by a constant force. Calculate the force.

$$\begin{aligned} \text{From 1.5(4):} \quad \bar{v}(4) - \bar{v}(0) &= \int_0^4 \bar{a}dt \\ 4\hat{x} - 6\hat{x} &= \int_0^4 \bar{a}dt = \bar{a}t|_0^4 = 4\bar{a} \\ \text{so that} \quad \bar{a} &= -2\hat{x}/4 = -0,5\hat{x} \quad \mathrm{m\,s^{-2}} \\ \text{and} \quad \bar{F} &= m\bar{a} = 12(-0,5\hat{x}) = -6\hat{x} \quad \mathrm{N} \end{aligned}$$

2. The mass of a motor car is 2000 kg and it has a speed of 72 $\mathrm{km\,h^{-1}}$ along a straight line over a horizontal surface. Calculate the force that the braking

system has to exert on the car to stop it over a distance of 100 m. Assume that the force is constant. Choose $\hat{x}$ in the direction of the velocity.

$$\begin{aligned}
72 \quad \mathrm{km\,h^{-1}} &= \frac{72\,000}{3600} = 20 \quad \mathrm{m\,s^{-1}} \\
\text{But} \qquad a_x\,dx &= v_x\,dv_x \\
\text{from which follows} \qquad \int_0^{100} a_x\,dx &= \int_{20}^{0} v_x\,dv_x \\
a_x x|_0^{100} &= \frac{1}{2} v_x^2|_{20}^{0} \\
100 a_x &= -200 \\
\text{so that} \qquad \bar{a} &= a_x \hat{x} = -2\hat{x} \quad \mathrm{m\,s^{-2}} \\
\text{and} \qquad \bar{F} &= m\bar{a} = 2000(-2\bar{x}) = -4000\hat{x} \quad \mathrm{N}
\end{aligned}$$

3. An aircraft with mass 5000 kg flies at constant velocity $100\hat{x}$ $\mathrm{m\,s^{-1}}$ when it enters a region of high humidity which causes ice to form on its wings and fuselage at a constant rate of 100 kg every 8 seconds. Calculate the extra force that the engines need to exert on the craft to keep its velocity constant. Disregard loss of mass due to fuel consumption. (Comment: Before the formation of ice, the aircraft moved force-free. The weight is balanced by the lift of the wings and the frictional drag by the forward thrust of the engines.)

Because the rate of ice formation is constant, it may be written:

$$\frac{dm}{dt} = \frac{\Delta m}{\Delta t} = \frac{100}{8} = 12,5 \quad \mathrm{kg\,s^{-1}}$$

From Newton's second law we have:

$$\begin{aligned}
\bar{F} = \frac{d\bar{p}}{dt} = \frac{d}{dt} m\bar{v} &= \frac{dm}{dt}\bar{v} + m\frac{d\bar{v}}{dt} = \frac{dm}{dt}\bar{v} \quad \text{because} \quad \bar{a} = \frac{d\bar{v}}{dt} = \bar{0} \\
\text{so that} \qquad \bar{F} &= 12,5 \times 100\hat{x} = 1,25 \times 10^3 \hat{x} \quad \mathrm{N}
\end{aligned}$$

4. A mass of 5 kg has a weight of 50 N and it is suspended by a light cord. Calculate the tension in the cord if the mass accelerates at 2 $\mathrm{m\,s^{-2}}$ (a) upwards, (b) downwards.

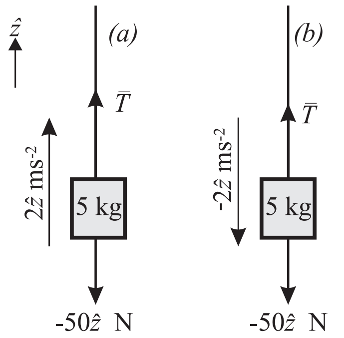


Figure 2.2-1

(a) The mass of the body is constant. Choose a frame of reference with $\hat{z}$ vertically upwards. Then:

$$\begin{aligned} \bar{F} &= m\bar{a} \\ \bar{T} - 50\hat{z} &= 5(2\hat{z}) \quad \text{so that} \\ \bar{T} &= 60\hat{z} \quad \text{N} \end{aligned}$$

(b) As above:

$$\begin{aligned} \bar{T} - 50\hat{z} &= 5(-2\hat{z}) \quad \text{so that} \\ \bar{T} &= 50\hat{z} - 10\hat{z} \\ &= 40\hat{z} \quad \text{N} \end{aligned}$$

2.2 The forces of nature

As far as is known today, only four interactions exist in nature. They are **gravitation, electromagnetic, weak nuclear interaction** and **the strong nuclear interaction**. In accordance with everyday experience, it is possible to list many other so-called "forces", but they can all be explained in terms of the four listed above. The existence of the nuclear forces become relevant only when the study of nuclear physics is involved.

All four forces describe interactions between material entities over a distance, i.e. they do not have to be in contact for the interaction to take place. Gravitation acts between mass and mass and is always an attractive force. Electromagnetic forces only act between electric charges. Electric charge is an inherent property of all matter and two kinds are known to exist. They are named **positive charge** and **negative charge** respectively because they have the property that they can neutralise the effects of each other. The connection between electric charge and matter becomes clear once one studies the structure of matter (molecule, atom, nucleus, proton, neutron, electron) in greater detail. For the sake of simplicity the term *electrostatic force* is used when no motion exists between charges. When charges move relative to each other, it is necessary to introduce the so-called magnetic force. For this reason it is convenient to use the term *electromagnetic interaction*. The interaction may be attractive or repulsive or it may have a direction with fits neither of these descriptions. Both gravity and the electromagnetic interaction are effective over long distances and are called **long-range forces**. Nuclear forces act only at distances less than a fixed value, and are known as **short-range forces** and represent the interaction between nucleons (proton-proton and proton-neutron). The strong nuclear force is by far the strongest interaction known, but it becomes effective only when the distance between two nucleons is in the order of 10^{-15} m.

Gravitation	1
Weak interaction	10^{25}
Electrostatic	10^{36}
Strong interaction	10^{38}

Table 2.2-1

Not much is known about the weak nuclear force and it was discovered in the study of the decay of radioactive atomic nuclei. The two nuclear forces are relevant only in the study of nuclear physics, and will not be discussed again in this text. Table 2.2-1 gives the relative magnitudes of these four forces between two protons which are near enough to enable the nuclear forces to be effective.

2.2.1 Gravitation

The attractive forces between *point masses* 1 and 2 with masses m_1 and m_2 respectively, are given by **Newton's law of gravitation**. Consider two such masses as shown in figure 2.2-2. Choose the origin of a frame of reference at point mass 1 and the position vector of 2, is represented by $\bar{r}$. The force which particle 1 exerts on 2, is indicated by $\bar{F}_{12}$ and that exerted by 2 on 1, by $\bar{F}_{21}$. Newton's law of gravitation is as follows:

$$\bar{F}_{21} = -\bar{F}_{12} = G\frac{m_1 m_2}{r^2}\hat{r} \qquad 2.2(1)$$

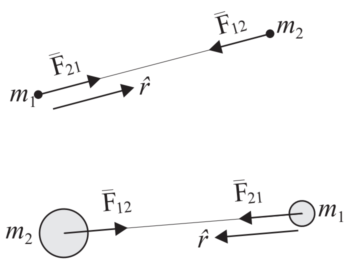


Figure 2.2-2

in which the **universal gravitational constant**, G, was measured by Cavendish. It is equal to $6,672 \times 10^{-11}$ $\mathrm{N\,m^2\,kg^{-2}}$. Cavendish's remarkable experiment showed that Newton's gravitational law applies also to spherical bodies as if point masses are situated at their centres. In the case of spheres (rather than point masses), the vector $\bar{r}$ is the position of the centre of mass of the one, relative to that of the other. Figure 2.2-2 shows the quantities which are referred to in equation 2.2(1), first for two mass points and then for two spheres.

To calculate the gravitational forces between bodies other than point masses and spheres, is outside the scope of this textbook. If such calculations have to be made, Newton's law for point masses is used. For two infinitely long thin straight parallel homogeneous cylinders, the forces are given by

$$\bar{F}_{21} = -\bar{F}_{12} = G\frac{\mu_1\mu_2}{r}\hat{r} \qquad 2.2(2)$$

in which μ_1 and μ_2 represent the masses per unit length of the two cylinders respectively and r the distance between their axes. Equation 2.2(2) is valid for the case where the origin of the frame of reference is on the axis of cylinder 1. It can further be shown that the forces between two infinitely large parallel plane mass distributions, are independent of the distance between them. Such mass distributions, of course, do not exist in nature and the results are only of academic interest. If the distance between the centres of mass of two bodies is large compared to their dimensions, they may be approximated by point masses and Newton's law will describe the forces that they exert on each other, very well.

The **weight** of a body near the Earth, is the force with which the Earth attracts it. Let M represent the mass of the Earth and m that of an object above its surface at a distance r from its centre (which is of course larger than the average radius of the Earth). The shape of the Earth does not differ much from a sphere and the size of the object is small compared to it. Newton's law thus applies to a high degree of accuracy. If the weight of the body is $\bar{W}$, it may be written that:

$$\bar{W} = -(GMm/r^2)\hat{r} \qquad 2.2(3)$$

in which $\hat{r}$ is a unit vector pointing away from Earth's centre of mass. By rearranging the factors in this equation, it becomes:

$$\bar{W} = m(-GM/r^2)\hat{r}$$

According to Newton's second law of motion, this force will accelerate the mass m. If the acceleration is represented by $\bar{g}$, Newton's second law may be written as follows:

$$\bar{W} = m\bar{g} \qquad 2.2(4)$$

From the preceding, it can be seen that

$$\bar{g} = (-GM/r^2)\hat{r} \qquad 2.2(5)$$

Equation 2.2(5) describes the gravitational acceleration, $\bar{g}$, in terms of other fundamental quantities. The Earth is approximately spherical with a radius of $R = 6,37 \times 10^6$ m and a mass of $M = 5,98 \times 10^{24}$ kg. If these values, together with $G = 6,67 \times 10^{-11}$ $\mathrm{N\,m^2\,kg^{-2}}$, are used in this equation, it follows that $\bar{g} = -9,83\hat{r}$ $\mathrm{m\,s^{-2}}$, which compares well with values measured experimentally on the surface of the Earth. The value of g is, of course, not constant on the Earth's surface. It is influenced by numerous factors such as the distance from the centre of mass of the Earth, the latitude (the rotation of the Earth about its axis), tides (the gravitational influence of the Sun and the Moon) and local variations in the density of the Earth's crust. Geologists use very accurate measurements of g to determine the presence of subterranean ore bodies. Geologists prefer the units **gal** and **milligal** when measuring g. 1 gal = 1 $\mathrm{cm\,s^{-2}}$ and it is named after Galileo Galilei, the first person to show that all objects have the same gravitational acceleration. The approximate value $g \approx 10\ \mathrm{m\,s^{-2}}$ is often used when numerical problems are solved.

Gravitational interaction may also be described by a different concept namely the **gravitational field**. Qualitatively the gravitational field of a given body is the region in which it will exert a gravitational force on another body. This implies that the gravitational field of every body is infinitely large, but since the gravitational effect of a point mass on another point mass varies as the inverse square of the distance between them, the force may usually be disregarded if the distance is large enough.

Qualitatively the gravitational field is described by the **gravitational field strength**. This is a vector quantity which depends on position and is defined as the force per unit mass at the position in question. The units are $\mathrm{N\,kg^{-1}}$, and the dimensions, $[\mathrm{L\,T^{-2}}]$. The latter is identical to that of acceleration.

The weight of a body with mass m in the gravitational field of the Earth, is given by:

$$\bar{W} = m\bar{g} = m(-GM/r^2)\hat{r} \quad \text{newton}$$

in which all the symbols have the same meaning as in equation 2.2(3). The force which a unit mass will experience at the same position, is the gravitational field strength which is given by:

$$\bar{W}/m = \bar{g} = -(GM/r^2)\hat{r} \quad \text{newton per kilogram} \qquad 2.2(6)$$

This expression is valid at position $\hat{r}$ *above* the Earth's surface and is based on the assumption that it is a homogeneous sphere. Far away from Earth, this expression, which is also exactly correct for the gravitational field strength of a point mass, is a good approximation. The gravitational field strength of the Earth in a vertical hole, does not obey the inverse square law.

It is interesting to note that gravitational field strength and the resulting gravitational acceleration at the same position, only differ in units. The formulas which describe them and also their dimensions are identical. The practical implication is that if a body is placed at a position where the gravitational field strength is equal to $\bar{a}\ \mathrm{N\,kg^{-1}}$, it will experience an acceleration of $\bar{a}\ \mathrm{m\,s^{-2}}$.

Examples

1.

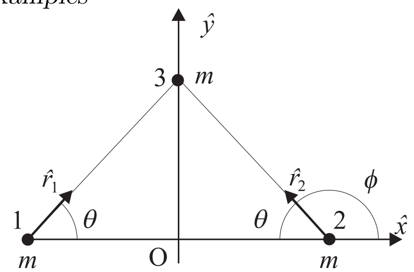


Figure 2.2-3

Two point masses of m kg each are on the x-axis of a Cartesian frame of reference at $x = a$ and $x = -a$ metre respectively. A third point mass of m kg is at position $\bar{r} = y\hat{y}$ metre. Calculate the force the first two masses exert on the third.

Let the two masses on the x-axis be 1 and 2 respectively and that on the y-axis, 3 as shown in figure 2.2-3.

From equation 2.2(1) follows:

$$\begin{aligned} \bar{F}_{13} &= -(Gmm/r^2)\hat{r}_1 \\ &= -(Gm^2/r^2)([\cos\theta]\hat{x} + [sin\theta]\hat{y}) \\ \bar{F}_{23} &= -(Gm^2/r^2)\hat{r}_2 \\ &= -(Gm^2/r^2)([\cos\phi]\hat{x} + [sin\phi]\hat{y}) \\ &= -(Gm^2/r^2)([-\cos\theta]\hat{x} + [sin\theta]\hat{y}) \quad \text{because} \quad \theta + \phi = \pi \\ \text{so that} \quad \bar{F} &= \bar{F}_{13} + \bar{F}_{23} = -(Gm^2/r^2)(2\sin\theta)\hat{y} \end{aligned}$$

But r and θ are not specified in the problem and must be expressed in terms of those quantities which are given (a and y). From the sketch it should be clear that:

$$\begin{aligned} r^2 &= a^2 + y^2 \qquad \text{and} \qquad \sin\theta = y/r = y/(a^2 + y^2)^{\frac{1}{2}} \quad \text{so that} \\ \bar{F} &= -\frac{2Gm^2 y}{(a^2 + y^2)^{\frac{3}{2}}}\hat{y} \qquad \text{newton} \end{aligned}$$

Problem: Show that the y-component of $\bar{F}$ has extreme values where $y = \pm 0,7071a$. Test each position to determine whether it is a maximum or a minimum. Draw a graph of the y-component of $\bar{F}$ as a function of y.

2. During the first manned mission to the Moon, a radio announcer was heard to have said that the spacecraft had reached the "half-way" mark to the Moon where "it leaves the gravitational field of Earth and enters that of the Moon." What he actually meant was that the spacecraft had reached the position where the vector sum of the two gravitational fields is zero. Assume that the mass of Earth is 80 times that of the Moon and that the distance between their centres of mass is $3,8 \times 10^8$ m. Calculate the position of this so-called half-way mark.

At the position in question, the magnitudes of the two fields are equal. Let M represent the mass of Earth and m that of the Moon. The magnitudes of the two gravitational fields are as follows:

$$\begin{array}{rrcl} \text{The Moon} & g_m & = & Gm/r^2 \\ \text{Earth} & g_e & = & GM/R^2 \end{array}$$

in which R is the distance from the centre of mass of the Earth, and r that from the centre of mass of the Moon. At the required position, the following statement is valid:

$$\begin{array}{lrcl} & GM/R^2 & = & Gm/r^2 \\ \text{from which follows} & R^2/r^2 & = & M/m = 80 \\ \text{so that} & R/r & = & (80)^{\frac{1}{2}} = 8,944 \\ \text{but} & R + r & = & 3,80 \times 10^8 \quad \text{m} \\ \text{and} & R & = & 3,42 \times 10^8 \quad \text{m} \end{array}$$

This distance is measured from the centre of mass of Earth and not from its surface. The spacecraft completes about 90% of its journey to reach this position. In the calculation the effect of the Sun was not taken into account.

2.2.2 The electromagnetic interaction

Electric charge is measured in coulomb (C). Consider two electric point charges 1 and 2 with magnitudes q_1 and q_2 respectively. Choose the origin of a frame of reference on charge 1 and let the position vector of the other charge be $\bar{r}$. The forces between the charges are given by **Coulomb's law**:

$$-\bar{F}_{21} = \bar{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{r} \qquad 2.2(7)$$

in which F is measured in newton and r in metre. In SI units the constant is $k = 8,98742 \times 10^9$. When a high degree of accuracy is not required, the approximation $k \approx 9 \times 10^9$ may be used.

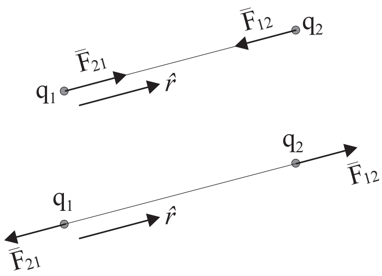


Figure 2.2-4

Superficially, Coulomb's law seems identical to Newton's gravitational law. Compare equations 2.2(1) and 2.2(7). A very important difference between them is that the signs of the forces are opposite. The reason is that positive and negative electric charges exist and that charges of equal sign repel each other whilst charges of opposite sign, attract. If charges 1 and 2 have the same sign, they repel and $\bar{F}_{12}$ is parallel to $\hat{r}$ with $\bar{F}_{21}$ in the opposite direction. If their signs differ, $\bar{F}_{21}$ is parallel to $\hat{r}$ with $\bar{F}_{12}$ in the opposite direction. Only positive mass exists and gravitational forces are always attractive.

In the dedicated study of electricity, it will be shown that equation 2.2(7) also describes the forces between two homogeneously charged spherical bodies. In that case, r is the distance between the centres of the spheres. For other **charge distributions**, Coulomb's law may be used to calculate the forces if the shapes of the charged bodies are known and also if the **charge densities** are either constant or known as functions of position. In this way it is possible to show that the magnitudes of forces between two infinitely long thin parallel homogeneously charged linear conductors, are inversely proportional to the distance between them. This configuration does not obey the inverse square law which applies to point charges and charged spheres.

The magnitudes of the forces between two infinitely large flat parallel homogeneous charge distributions, depend only upon the **surface charge densities**, certain properties of the material which separates them, and not the distance between them.

The **electric field strength**(also known as the **electric field intensity**) at a given position is defined as the electric force which a unit positive charge will experience when placed there. Electric field strength is a vector field (i.e. a vector function of position) and is measured in newton per coulomb ($\mathrm{N\,C^{-1}}$).

The electric field strength, $\bar{E}$ at position $\bar{r}$ from a point charge q, may be calculated directly from Coulomb's law and is given by:

$$\bar{E} = (kq/r^2)\hat{r} \qquad 2.2(8)$$

Some prefer to write $\bar{E} = (kq/r^3)\bar{r}$ which is exactly the same as equation 2.2(8) since $\hat{r} = \bar{r}/r$. The electrostatic field of a single point charge and the gravitational field of a single point mass, are both **radial fields** with **spherical symmetry**. In both cases the magnitude decreases as the inverse square of the distance from the point. For this reason Newton's law of gravitation and the Coulomb law are known as **inverse square laws**. In the case of a positive point charge, the field vector points radially away from the charge and in the case of a negative point charge, towards the charge.

The electrostatic field is a powerful concept for the description of electromagnetic phenomena. One of the uses is the calculation of the force, $\bar{F}$, which a point charge, Q, will experience in an electric field $\bar{E}$. The relationship is

$$\bar{F} = Q\bar{E} \qquad 2.2(9)$$

The forces between charges which are in motion relative to each other, are described by means of **magnetic fields** which will be treated fully during the study of electricity and magnetism.

Examples:

1. A deuteron is the nucleus of a ${}^2_1$H-atom (heavy hydrogen or deuterium.) Its mass is $3,34 \times 10^{-27}$ kg and its electric charge, $1,60 \times 10^{-19}$C. The proportionality constant in Coulomb's law is $k = 8,99 \times 10^9$ N m^{2}C^{-2} and that in Newton's gravitational law, $G = 6,67 \times 10^{-11}$ N m^{2}kg^{-2}. The magnitude of the gravitational force between two deuterons is represented by F_g, and that of the electrostatic force, F_e. Calculate F_e/F_g.

$$\begin{aligned} &\text{From 2.2(1):} \qquad & F_g &= Gm^2/r^2 \\ &\text{From 2.2(7):} \qquad & F_e &= kq^2/r^2 \end{aligned}$$

in which m represents the mass of a deuteron and q its electric charge. The distance between them is r. Then

$$\begin{aligned} F_e/F_g = kq^2/Gm^2 &= \frac{8,99 \times 10^9 \times (1,60 \times 10^{-19})^2}{6,67 \times 10^{-11} \times (3,34 \times 10^{-27})^2} \\ &= 3,09 \times 10^{35} \end{aligned}$$

If one bears in mind that the visible universe has a diameter of about 10^{30} mm, the immensity of the electrical force compared to the gravitational force should be evident.

2. A positive electric point charge of $+q$ coulomb is at position $x = -a$ metre on the x-axis of a Cartesian frame of reference and a negative point charge at $x = +a$ metre. Calculate the electric field strength at position $\bar{r} = y\hat{y}$ which is caused by these two charges. (Comment: Two electric point charges with equal magnitude and opposite sign which do not coincide, constitute a so-called **electric dipole** of which the **electric dipole moment** is given by $\bar{p} = q\bar{\ell}$. The

units of $\bar{p}$ are coulomb metre (C m) and $\bar{\ell}$ is the position of the positive charge relative to the negative, i.e. the vector $\bar{\ell}$ begins at the negative charge and ends at the positive. (See solution for further clarification.)

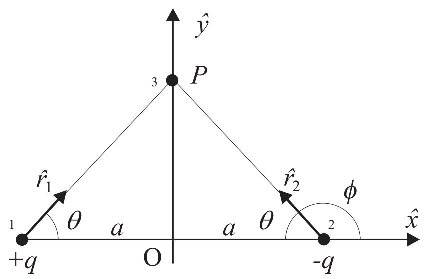


Figure 2.2-5

Let the electric field strength at P which is caused by $+q$ be represented by $\bar{E}_{1P}$, and that caused by $-q$, $\bar{E}_{2P}$. The corresponding unit vectors from the two charges towards P, are $\hat{r}_1$ and $\hat{r}_2$ respectively. See figure 2.2-5 for further explanation.

From 2.2(8) follows:

$$\begin{aligned}
\bar{E}_{1P} &= (k[+q]/r^2)\hat{r}_1 \\
&= (kq/r^2)([cos\theta]\hat{x} + [sin\theta]\hat{y}) \\
\bar{E}_{2P} &= (k[-q]/r^2)\hat{r}_2 \\
&= (-kq/r^2)([cos\phi]\hat{x} + [sin\phi]\hat{y}) \\
&= (-kq/r^2)([-cos\theta]\hat{x} + [sin\theta]\hat{y}) \quad \text{because} \quad \theta + \phi = \pi
\end{aligned}$$

The resultant field strength is given by

$$\bar{E} \quad = \quad \bar{E}_{1P} + \bar{E}_{2P} = (kq/r^2)(2\cos\theta)\hat{x}$$

But r and $\cos\theta$ are not specified in the problem and have to be expressed in terms of a and y which are given. From figure 2.2-5 it can be seen that:

$$\begin{aligned}
r^2 &= a^2 + y^2 \qquad \text{and} \qquad \cos\theta = a/r = a/(a^2+y^2)^{\frac{1}{2}} \qquad \text{so that} \\
\bar{E} &= \frac{2kqa\hat{x}}{(a^2+y^2)^{\frac{3}{2}}} \quad \mathrm{N\,C^{-1}}
\end{aligned}$$

The position vector of the positive charge relative to the negative, is given by $\bar{\ell} = -2a\hat{x}$, so that $\bar{p} = -2aq\bar{x}$ C m. Expressed in terms of the dipole moment, the electrostatic field strength at P is given by

$$\bar{E} = \frac{-k\bar{p}}{(a^2+y^2)^{\frac{3}{2}}}$$

Comment: The directions of the vectors $\bar{E}_{1P}$ and $\bar{E}_{2P}$ are not shown in figure 2.2-5 to indicate to readers how the definition of field strength which is the result of a point charge, is used in a calculation. Beginners find it difficult to establish the direction of the unit vector, $\hat{r}$, which always points away from the point charge, irrespective of its sign.

3.

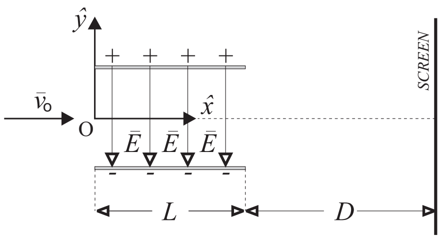


Figure 2.2-6

The mass of an electron is m kilogram and its electric charge, $-e$ coulomb. It has an initial velocity of $\bar{v}(0) = v_0\hat{x}$ $\mathrm{m\,s^{-1}}$ when it enters the electric field between the deflection plates of a cathode-ray tube. The choice of the axes is explained in figure 2.2-6. Choose $t = 0$ when the electron enters the field at the origin. Between the plates the electrostatic field is $\bar{E} = -E\hat{y}$ $\mathrm{N\,C^{-1}}$. The length of each of the parallel plates is L metres and in the region not between them, the electrostatic field is zero. The fluorescent screen is D metres from the deflection plates. Calculate the velocity and position of the electron between the plates as functions of time. Then calculate the Cartesian equation of its path and the angle that it makes with $\hat{x}$ when the electron leaves the field. Finally, calculate the position where the electron strikes the screen.

The choice of the axes, makes the problem two-dimensional. The force on the electron may be calculated by using 2.2(9).

$$\bar{F} = (-e)(-E\hat{y}) = eE\hat{y} \qquad \text{newton}$$

The mass of the electron is m kilogram and its acceleration follows from Newton's second law.

$$\bar{a} = \bar{F}/m = (eE/m)\hat{y} = 0\hat{x} + (eE/m)\hat{y} \quad \mathrm{m\,s^{-2}}$$

The problem is solved in the same way as all other problems on projectiles which were treated in 1.7.

$$\begin{aligned}
\bar{v} &= \int \bar{a}dt = \int (0\hat{x} + [eE/m]\hat{y})dt \\
&= v_0\hat{x} + (eE/m)t\hat{y} \quad \mathrm{m\,s^{-1}} \qquad\qquad \text{because} \quad \bar{v}(0) = v_0\hat{x} \\
\bar{r} &= \int \hat{v}dt = \int (v_0\hat{x} + [eE/m]t\hat{y})dt \\
&= (v_0t)\hat{x} + [\frac{1}{2}(eE/m)t^2]\hat{y} \quad \mathrm{m} \qquad\qquad \text{because} \quad \bar{r}(0) = \bar{0}
\end{aligned}$$

The path of the electron is given by the following parametric equations:

$$x = v_0t \qquad \text{and} \qquad y = \frac{1}{2}(eE/m)t^2$$

Elimination of t gives the Cartesian equation for the trajectory.

$$y = \frac{1}{2}(eE/m)(x/v_0)^2 = \frac{1}{2}(eE/mv_0^2)x^2$$

The gradient of the trajectory is given by the derivative of its equation:

$$dy/dx = (eE/mv_0^2)x = \tan\alpha$$

in which α is the angle between the tangent (or direction of the velocity vector) and $\hat{x}$. At the position where the electron leaves the electric field, $x = L$. At this position the gradient of the trajectory is thus given by:

$$dy/dx|_{x=L} = LeE/mv_0^2$$

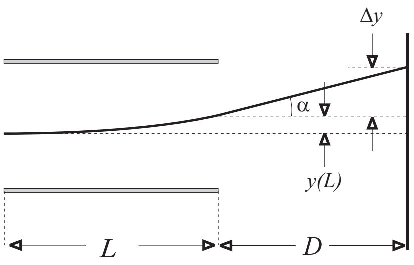


Figure 2.2-7

At this stage of its flight, the electron leaves the electric field (here $x = L$), and its deviation from the x-axis is given by:

$$y(L) = \frac{1}{2}(eE/mv_0^2)L^2$$

Once it has left the field, it moves force-free and thus along a straight line according to Newton's first law. The additional deviation outside the deflection field, is given by:

$$\Delta y = D\tan\alpha = D(LeE/mv_0^2) \qquad \text{(See figure 2.2-7)}$$

in which the value of $\tan\alpha$ is used where $x = L$ as it was calculated above. The position where the electron strikes the screen is now given by

$$\begin{aligned} y &= y(L) + \Delta y \\ &= \frac{1}{2}(eE/mv_0^2)L^2 + D(LeE/mv_0^2) \\ &= (LeE/mv_0^2)(L/2 + D) \end{aligned}$$

This result is of importance in the use of an oscilloscope.

2.3 Friction

When two solid bodies are in contact, a force which tends to oppose their relative motion or pending relative motion, exists at the contact surface. If a horizontal force is applied to a heavy object resting on a floor, it will not move until this opposing force is overcome by the applied force. If an object moves across a horizontal surface with a given initial velocity and with no other forces applied to it, it will finally come to rest because of this opposing force. This **contact force** is always in the direction opposite to the existing or pending motion. The phenomenon is called **friction**.

Frictional forces also exist between solids and fluids (liquid and gas). If the flow of a fluid is to be maintained in a pipe, a difference of pressure is required between its two ends. This friction is caused by the **viscosity** of the fluid. In this section friction between solids only will be studied.

The frictional force between two smooth solid bodies is the macroscopic result of the interactions between the molecules in one body and those in the other. It is impossible to study frictional force as the resultant of all these microscopic interactions. For practical purposes, it is sufficient that it is studied **phenomenologically**. The following experimental results are sufficient for the solution of most problems which deal with friction between two solid objects.

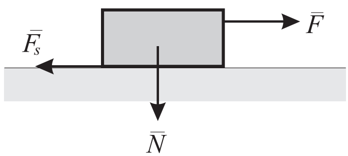


Figure 2.3-1

Consider a body resting on a solid horizontal plane as shown in figure 2.3-1. If $\bar{F}$ is the force with the least magnitude required to set the body in motion, it is equal in magnitude and opposite in direction to the frictional force, $\bar{F}_s$. Experimentally it has been found that:

$$F = F_s = \mu_s N \qquad 2.3(1)$$

in which N is the magnitude of the normal (perpendicular) force which acts between the body and the surface and μ_s is called the **coefficient of static friction** and which is a constant for the two **materials** which are in contact. Equation 2.3(1) is not a vector equation; it only describes the relationship between the magnitudes of two forces.

If the body is in motion and slides across the horizontal plane, a similar relationship describes the frictional force:

$$F_k = \mu_k N \qquad 2.3(2)$$

in which F_k is the magnitude of the frictional force and N the magnitude of the normal (perpendicular) force between the body and the surface on which it is sliding, and μ_k, is the **coefficient of kinetic friction** which is also a constant for the two materials in contact. The two coefficients of friction usually differ for a pair of given materials, μ_k usually being less than μ_s.

Materials	μ_s	μ_k
teflon on teflon	0,04	0,04
teflon on steel	0,04	0,04
brass on steel	0,52	0,45
copper on steel	0,53	0,35
steel on steel	0,78	0,42

Table 2.3-1

In table 2.3-1 the values of the two coefficients of friction are given for a few pairs of materials. The values apply only to "clean" and "smooth" surfaces. The words *clean* and *smooth* are taken to mean that no observable other materials are present and that the surfaces appear smooth when visually inspected and feel smooth when touched.

Consider a body of mass m kg on an inclined plane of which the inclination angle is θ. It is in such a condition that if the angle is increased by the slightest amount, the body will start sliding down the plane. The component of the weight normal to the plane is $mg\cos\theta$ and it is this component that presses the body against the plane. The component of the weight parallel to the plane is $mg\sin\theta$ and this component tends to accelerate the body downwards along the incline. If motion is imminent, $mg\sin\theta$ will be equal to the static frictional force in the opposite direction.

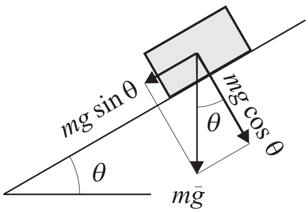


Figure 2.3-2

$$mg\sin\theta = \mu_s mg\cos\theta$$

$$\text{so that} \qquad \mu_s = \frac{\sin\theta}{\cos\theta} = \tan\theta \qquad 2.3(3)$$

This critical angle at which sliding is imminent, is known as the **angle of friction**. The experiment for which the calculation was made, presents a simple but efficient way to measure the coefficient of static friction between two materials. The body is placed on the plane of which the inclination angle can be varied. The magnitude of the angle when sliding is imminent, can be determined experimentally to a surprising degree of accuracy. The calculation is made by means of equation 2.3(3).

Consider the same mass on an inclined plane but with the inclination angle larger than the angle of friction. The component of the weight parallel to the plane is greater than the frictional force and the mass will accelerate downwards. The resultant force downwards along the plane is given by:

$$F = mg\sin\theta - \mu_k N = mg\sin\theta - \mu_k mg\cos\theta$$

This force accelerates the body along the plane in accordance with Newton's second law:

$$F = ma = mg(\sin\theta - \mu_k \cos\theta)$$

and the acceleration parallel to the plane is given by:

$$a = g(\sin\theta - \mu_k \cos\theta)$$

Examples:
1. A body of 200 kg rests on a horizontal plane. The coefficient of static friction between the mass and the plane, is 0,5. F is the minimum magnitude of the force required to set the body in motion. Calculate F if its direction is (a) horizontal, (b) 30° upwards from the horizontal plane, (c) 30° downwards from the horizontal plane. The three options are shown in figure 2.3-3.

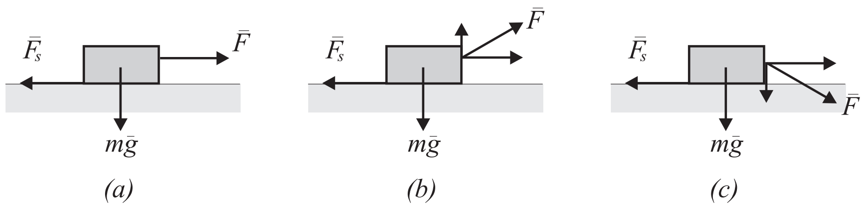


Figure 2.3-3

(a) The magnitude of the body's weight is $mg = 200 \times 10 = 2000$ N. The weight is the only force which presses the body against the plane, and the frictional force is given by:

$$F_s = \mu_s N = 0,5 \times 2000 = 1000 \quad \text{N}$$

Motion will be imminent if the applied force is equal to the frictional force. Therefore

$$F = 1000 \quad \text{N}$$

(b) The component of the applied force, F, normal to the horizontal plane, is equal to $F \sin 30°$. The force pressing the body against the plane is less than in the previous problem and is given by:

$$N = 2000 - F \sin 30°$$

The frictional force is given by:

$$F_s = \mu_s N = \mu_s (2000 - F \sin 30°)$$

The component of the applied force parallel to the plane is $F\cos 30°$ and motion will be imminent if this force is equal to the frictional force. From this follows:

$$\begin{aligned} F\cos 30° &= \mu_s(2000 - F\sin 30°) \\ 0,8660F &= 0,5(2000 - 0,5000F) = 1000 - 0,2500F \\ F &= 896,06 \quad \text{N} \end{aligned}$$

(c) This problem is similar to the previous one with the exception that the vertical component of the applied force is in the same direction as the weight and thus contributes to the total force pressing the body against the plane.

$$\begin{aligned} 0,8660F &= 1000 + 0,2500F \qquad \text{so that} \\ F &= 1623,3 \quad \text{N} \end{aligned}$$

2. A body of unknown mass is in contact with an inclined plane of which the inclination angle is 30°. This angle is less than the frictional angle and the body will not slide down the plane by gravity alone. It is found that a downward force of 1 N parallel to the plane is required to initiate motion in that direction. If an upward force of 10 N parallel to the plane is applied, it will start moving in that direction. $g = 10$ m s^{-2}. Calculate the coefficient of friction between the body and the plane and also the mass of the body.

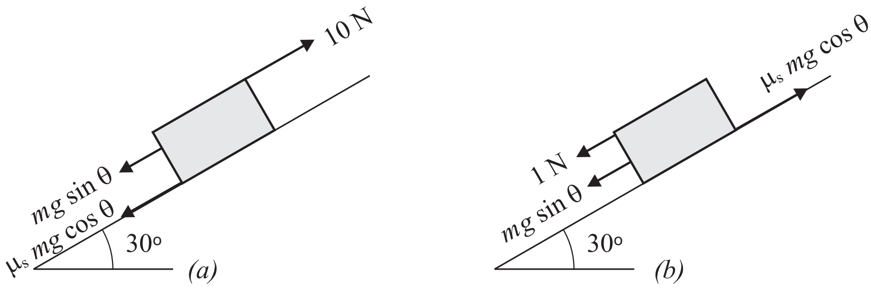


Figure 2.3-4

To start the body moving upwards along the plane, the applied force must be equal to the sum of the frictional force and the component of the weight as shown in figure 2,3-4(a).

$$10 = 10m\sin 30° + \mu_s 10m\cos 30°$$

To start the body moving downwards along the plane, the sum of the applied force and the component of the weight parallel to the plane, must be equal to the frictional force, as shown in figure 2.3-4(b)

$$1 + 10m\sin 30° = \mu_s 10m\cos 30°$$

The equations from which the unknown quantities may be calculated are as follows:

$$\begin{aligned} 8,660\mu_s m + 5m &= 10 \\ 8,660\mu_s m - 5m &= 1 \end{aligned}$$

from which follows $m = 0,9000$ kg and $\mu_s = 0{,}7057$.

3. A body of mass 3 kg is projected upwards along a rough inclined plane of which the angle of inclination is 15°. The initial speed is 15 m s^{-1}. The coefficient of kinetic friction is 0,2. Calculate (a) The frictional force whilst the body is moving upwards along the plane. (b) How far it will move before coming to rest. (c) How long it will take to complete the upward motion. (d) At what speed it will arrive at the position from which it was projected upwards along the plane, when sliding back from its position of maximum height. (e) How long it will take to return to the initial position. $g = 10$ m s^{-2}.

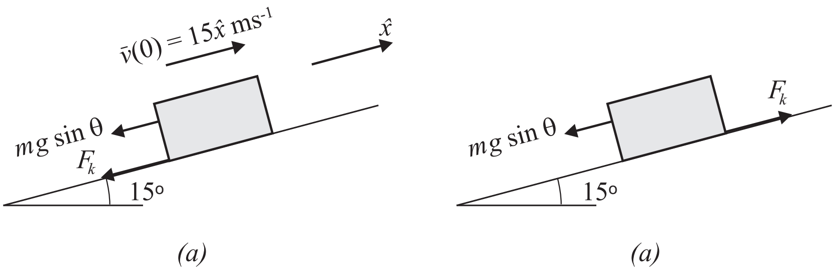


Figure 2.3-5

Choose a Cartesian frame of reference with origin at the lower end of the inclined plane and $\hat{x}$ parallel to it as shown in figure 2.3-5. (a) The magnitude of the frictional force is given by:

$$\begin{aligned} F_k &= \mu_k mg\cos\theta \\ &= 0,2 \times 3 \times \cos 15^\circ \\ &= 5,796 \quad \text{N} \end{aligned}$$

(b) While the body is moving upwards, the resultant force on it parallel to the plane is:

$$\begin{aligned} \bar{F}' &= (-mg\sin\theta)\hat{x} + (-F_k)\hat{x} \\ &= -7,765\hat{x} - 5,796\hat{x} \\ &= -13,561\hat{x} \quad \text{N} \end{aligned}$$

Under the action of this force, the body experiences an acceleration.

$$\begin{aligned} \bar{a}' &= \bar{F}'/m = (-13,561\hat{x})/3 = -4,520\hat{x} \quad \mathrm{m\,s^{-2}} \\ \text{but} \quad a_x dx &= v_x dv_x \\ \text{and} \quad \int_0^x -4,520 dx &= \int_{15}^0 v_x dv_x \\ -4,520x|_0^x &= \frac{1}{2} v_x^2|_{15}^0 \\ x &= 24,89 \quad \mathrm{m} \end{aligned}$$

(c)

$$\begin{aligned} \Delta\bar{v} = \bar{v}(t) - \bar{v}(0) &= \int_0^t \bar{a} dt = \int_0^t -4,520\hat{x} dt \\ 0\hat{x} - 15\hat{x} &= -4.520t\hat{x} \\ \text{and} \quad t &= 3,319 \quad \mathrm{s} \end{aligned}$$

(d) The resultant force during the downward motion is given by:

$$\begin{aligned} \bar{F}'' &= (-mg\sin\theta)\hat{x} + (F_k)\hat{x} \\ &= -7,765\hat{x} + 5,796\hat{x} = -1,969\hat{x} \quad \mathrm{N} \end{aligned}$$

This force causes the following acceleration:

$$\begin{aligned} \bar{a}'' &= \bar{F}''/m = (-1,969\hat{x})/3 \\ &= -0,6563\hat{x} \quad \mathrm{m\,s^{-2}} \\ \text{As before} \quad \int_{24,89}^0 -0,6563 dx &= \int_0^v v_x dv_x \\ -0,6563x|_{24,89}^0 &= \frac{1}{2} v_x^2|_0^v \\ \text{from which follows} \quad v &= \pm 5,716 \quad \mathrm{m\,s^{-1}} \\ \text{so that} \quad \bar{v} &= -5,716\hat{x} \quad \mathrm{m\,s^{-1}} \end{aligned}$$

(e) The time is calculated in the same way as was done in (c).

$$\begin{aligned} \Delta\bar{v} = -5,716\hat{x} - 0\hat{x} &= \int_0^t -0,6563\hat{x} dt = -0,6563t\hat{x} \\ \text{so that} \quad t &= 8,709 \quad \mathrm{s} \end{aligned}$$

2.4 Work and power

2.4.1 Work

When a force acts on a body while it is being displaced, the force will, in general, do **work**. If a constant force $\bar{F}$ (both the magnitude and the direction remain constant) acts on a point mass while a displacement $\Delta\bar{r}$ occurs along a straight line parallel to the force, the work done is defined as follows:

$$A = |\bar{F}| \times |\Delta\bar{r}| = F \times |\Delta\bar{r}| \qquad 2.4(1)$$

in which $|\bar{F}| = F$ is the magnitude of the force and $|\Delta\bar{r}|$, the magnitude of the displacement. If the force specified in equation 2.4(1) has a magnitude of 1 newton and that of the displacement is 1 metre, the work done is 1 *joule*(J). It may thus be stated that:

$$1 \text{ J} = 1 \text{ N} \quad \times \quad 1 \text{ m}$$

which is valid only for the specific conditions previously described (force constant and displacement along a straight line parallel to the force vector). The dimensions of work are $[\text{A}] = [\text{M}\,\text{L}^2\text{T}^{-2}]$.

If the constant force, $\bar{F}$, acts at an angle θ to the displacement along a straight line, only the component of the force parallel to the displacement will do work as follows:

$$A = (F\cos\theta) \times (|\Delta\bar{r}|) = \bar{F} \cdot \Delta\bar{r} \qquad 2.4(2)$$

which, once again, is valid only for the specified conditions (force constant, and displacement along a straight line at angle θ to the force vector).

From equation 2.4(2), it can be seen that no work is done if the force and displacement vectors are orthogonal. If a person moves a mass vertically away from or towards the surface of the earth, work is done. When the same mass is carried by the person whilst it is kept at the same vertical height, no work is done. From equation 2.4(2), it can also be seen that the work done, may be negative or positive ($-1 \leq \cos\theta \leq 1$). If the work is positive, the work is done **by** the source of the force and if it is negative, the work is done **on** the source of the force by the system to which the mass belongs.

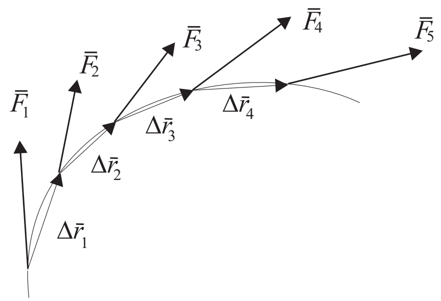


Figure 2.4-1

In general, the force will be a vector function of position (a vector field) which means that a different force will exist for each position on the path along which the mass moves. Furthermore, the path will, in general, not be a straight line. For such cases, equations 2.4(1) and 2.4(2) are not valid for the calculation of the work done.

In principle, the space curve along which the motion occurs, may be subdivided into a large number of small intervals, each of which will be small enough for the force to be taken as constant along it. If a total of n such intervals exists, the work done during displacement number i, may be approximated by the following:

$$\Delta A_i = \bar{F}_i \cdot \Delta\bar{r}_i$$

The smaller the intervals, the better the approximations will be. The total amount of work done, will be given by the limit of the sum over all the subintervals when their magnitudes tend towards zero. See figure 2.4-1.

$$A = \lim_{\Delta\bar{r}_i \to \bar{0}} \sum_{i=1}^{n} \bar{F}_i \cdot \Delta\bar{r}_i$$

But that is identical to the following integral equation:

$$A = \int_P^Q \bar{F} \cdot d\bar{r} \qquad 2.4(3)$$

along the space curve between the two points P and Q, which define the total displacement. P and Q do not represent limits of a definite integral in the normal sense; they symbolically represent the initial and final points of the displacement along the path. This integral is known as a **line integral** of the force vector between the two given positions *along the specified space curve.* In simple problems where the force vector is constant or if it consists of a single component in a given frame of reference *and* the displacement is along a straight line, the calculation of such an integral is very simple. In cases where the force vector is a function of position, and the space curve is known as a function of a scalar parameter, this parameter has to be used to calculate the work. In any case, it is useful to start each problem by writing the work in differential form. This form may, indeed, be taken as an alternative very useful definition of work.

$$dA = \bar{F} \cdot d\bar{r} \qquad 2.4(4)$$

in which

$$d\bar{r} = dx\hat{x} + dy\hat{y} + dz\hat{z} \qquad 2.4(5)$$

If the symbol s is used to measure the path length along the space curve, it may be written that

$$ds = |d\bar{r}| = [(dx)^2 + (dy)^2 + (dz)^2]^{\frac{1}{2}}$$

as was shown in equation 1.2(5) in 1.2.2. The vector $d\bar{s}$ is thus tangential to the space curve and identical in all respects to $d\bar{r}$. It may thus be written that

$$dA = \bar{F} \cdot d\bar{s} = F_s ds \qquad 2.4(6)$$

$$\text{so that} \quad A = \int_P^Q \bar{F} \cdot d\bar{s} = \int_P^Q F_s ds \qquad 2.4(7)$$

in which $F_s = F \cos\theta$ is the **tangential component** of the force, $\bar{F}$, (i.e. tangential to the space curve). The **normal component** which is also called the **centripetal component**, does no work.

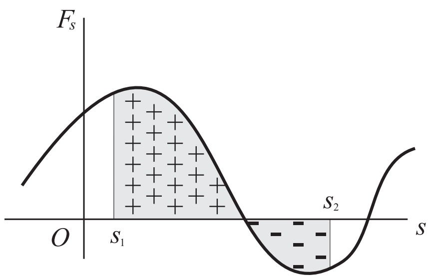


Figure 2.4-2

As is known from the theory of integration, the value of a definite integral may be represented by the *net* area under the graph of the integrand. If F_s is known as a function of s and it is represented as a graph, the work done will be given by the *net* area as shown in figure 2.4-2. In this graph the values s_1 and s_2 correspond to the positions P and Q referred to in 2.4(3) and 2.4(7). They are simply the values of the path length when the body is at the two given positions.

It was necessary to emphasise the word *net* to indicate that the work may be either positive or negative. From the theory of integration it is known that the area above the s-axis is positive, and that below it, negative.

Examples

1. A force $\bar{F} = 13\hat{z}$ N acts on a point mass during its displacement from the origin along the z-axis to the position $\bar{r} = 4\hat{z}$ m. Calculate the work done by the force.

$$dA = \bar{F} \cdot d\bar{r} = (13\hat{z}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z}) = 13dz$$

$$\text{and} \quad A = \int_0^4 13dz = 13z|_0^4 = 52 \quad \text{J}$$

Because the force is constant (it is not a function of position or time) and the displacement is along a straight line parallel to the force vector, it may be written directly that:

$$A = F \times \Delta r = 13 \times 4 = 52 \quad \text{J}$$

which gives exactly the same answer.

2. The mass of a helicopter including its contents, is 3000 kg. Calculate the work required to lift it 500 m vertically at a constant velocity. Disregard frictional drag. $g = 10\ \mathrm{m\,s^{-2}}$.

Choose a frame of reference with origin on ground level and $\hat{z}$ vertically upwards. The weight of the helicopter is $(-mg)\hat{z} = -3000 \times 10\hat{z} = -3 \times 10^4\hat{z}$ N. If it moves at constant velocity, it is force-free according to Newton's first law and the magnitude of the applied force is equal to that of the weight but in the opposite direction. (If the helicopter starts from rest, it will have to be accelerated to achieve its velocity and the magnitude of the applied force will have to be larger than that of the weight. This situation, however, is outside the scope of this problem.) The applied force is thus $\bar{F} = +3 \times 10^4\hat{z}$ N.

$$\begin{aligned} dA &= (3 \times 10^4\hat{z}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z}) = 3 \times 10^4 dz \\ \text{and} \qquad A &= \int_0^{500} 3 \times 10^4 dz = 1,5 \times 10^7\ \mathrm{J} = 15\ \mathrm{MJ} \end{aligned}$$

Due to the simplicity of this problem, it would have been much easier to use the definition in equation 2.4(1). The reason for using vector notation is to introduce readers to its proper use in the calculation of work.

3. A length of rubber band is to be stretched within its elastic limit. The magnitude of the force required, is equal to $200x$ N in which x represents the extension or elongation of the rubber from the condition where it was not deformed at all. The number 200, is called the **force constant** of the rubber band. It is the force per unit length of extension and is represented by gradient of the force-extension graph (i.e. dF/dx) and measured in $\mathrm{N\,m^{-1}}$.

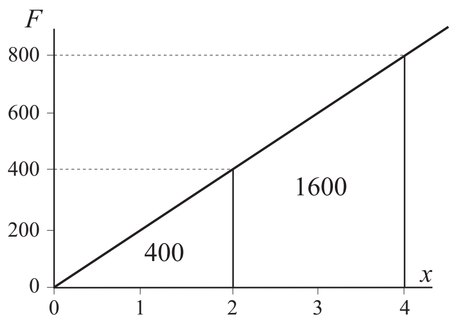


Figure 2.4-3

Choose a frame of reference with origin at the position of the free end when the extension is zero, with $\hat{x}$ downwards in the direction in which the extension is to take place. Calculate the work done by the applied force during the following extensions: (a) From $x = 0$ to $x = 2$ m, (c) form $x = 2$ to $x = 4$ m. Draw a graph of $F = F(x)$ and indicate the answers on it.

(a)

$$\begin{aligned} dA &= \bar{F} \cdot d\bar{r} = (200x\bar{x}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z}) \\ &= 200x dx \end{aligned}$$

$$A = \int_0^2 200x dx = 100x^2|_0^2 = 400 \quad \text{J}$$

(b) As in the previous problem:

$$A = 100x^2|_2^4 = 1600 - 400 = 1200 \quad \text{J}$$

It is interesting to note that the extensions (i.e. the magnitude of the displacements) are the same in (a) and (b). The work done, however, differs to a large extent because the magnitude of the force increases with increasing extension.

4. A and B are two electric charges. A is fixed at the origin of a Cartesian frame of reference and the motion of B is limited to the positive x-axis. B experiences a repelling force of $2x^{-2}\hat{x}$ N in which x is measured in metres. Calculate the work done by an applied force to move B at a constant velocity (a) from $\bar{r} = 4\hat{x}$ m to $\bar{r} = 2\hat{x}$ m, (b) from infinity to $\bar{r} = 2\hat{x}$ m, (c) from infinity to $\bar{r} = a\hat{x}$ m.

In order to move the body which carries charge B at a constant velocity, it has to be force-free (Newton's first law), and therefore the applied force must be equal in magnitude to the electrical force, but opposite in direction, i.e. $\bar{F} = (-2x^{-2})\hat{x}$ N.

(a)
$$dA = \bar{F} \cdot d\bar{r} = (-2x^{-2}\hat{x}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z}) = -2x^{-2}dx$$
$$A = \int_4^2 -2x^{-2}dx = 2x^{-1}|_4^2 = 2(\frac{1}{2}) - 2(\frac{1}{4}) = 0,50 \quad \text{J}$$

(b) As above:
$$A = 2x^{-1}|_\infty^2 = 2(\frac{1}{2}) - 2(\frac{1}{\infty}) = 1 - 0 = 1 \quad \text{J}$$

(c) As above:
$$A = 2x^{-1}|_\infty^a = 2(\frac{1}{a}) - 2(\frac{1}{\infty}) = 2/a \quad \text{J}$$

5. A body of mass 2 kg moves along a space curve. Its position vector is given by:

$$\bar{r} = (t^2 - 2t + 5)\hat{x} + (-t^3 + 3t)\hat{y} + (4t - 2)\hat{z} \quad \text{metres}$$

in which the time, t, is measured in seconds. Calculate the work done on the mass while it is moved by an applied force from position $\bar{r}_1 = 4\hat{x} + 2\hat{y} + 2\hat{z}$ m to $\bar{r}_2 = 5\hat{x} - 2\hat{y} + 6\hat{z}$ m.

While the body remains on the specified space curve, its velocity is given by:

$$\bar{v} = d\bar{r}/dt = (2t - 2)\hat{x} + (-3t^2 + 3)\hat{y} + (4)\hat{z} \quad \text{m s}^{-1}$$

and its acceleration by:

$$\bar{a} = d\bar{v}/dt = (2)\hat{x} + (-6t)\hat{y} + (0)\hat{z} \quad \text{m s}^{-2}$$

This acceleration is caused by a force which acts upon the body. According to Newton's second law, the force is given by:

$$\bar{F} = m\bar{a} = (4)\hat{x} + (-12t)\hat{y} + (0)\hat{z} \quad \text{N}$$

Using the velocity vector, an expression for $d\bar{r}$ may be found:

$$d\bar{r} = (d\bar{r}/dt)dt = (2t-2)dt\hat{x} + (-3t^2+3)dt\hat{y} + (4)dt\hat{z} \quad \text{m}$$

from which it can be seen that:

$$dx = (2t-2)dt \qquad dy = (-3t^2+3)dt \qquad dz = (4)dt$$

From this follows:

$$\begin{aligned} dA &= \bar{F}\cdot d\bar{r} \\ &= [(4)\hat{x} + (-12t)\hat{y} + (0)\hat{z}]\cdot[(2t-2)dt\hat{x} + (-3t^2+3)dt\hat{y} + (4)dt\hat{z}] \\ &= (8t-8)dt + (36t^3-36t)dt \\ &= (36t^3-28t-8)dt \end{aligned}$$

It is left as an exercise to the reader to show that the two specified positions, $\bar{r}_1$ and $\bar{r}_2$, actually lie on the space curve and that the mass will be there when $t = 1$ s and $t = 2$ s respectively. The work is given by:

$$\begin{aligned} A = \int_1^2 (36t^3-28t-8)dt &= 9t^4 - 14t^2 - 8t|_1^2 \\ &= 72 + 13 \\ &= 85 \quad \text{J} \end{aligned}$$

2.4.2 Power

Power is the quantity which measures the time-rate at which work is being done. It is necessary to distinguish between **average power over a given interval of time** and **instantaneous power** at a given instant.

The average power is calculated by dividing the total amount of work divided by the magnitude of the time interval as follows:

$$\text{average power} = P_{av} = \frac{\Delta A}{\Delta t} \qquad 2.4(8)$$

The instantaneous power is given by the rate at which work is being done at that instant.

$$\text{instantaneous power} = P = \frac{dA}{dt} \qquad 2.4(9)$$

If equation 2.4(9) is to be used to calculate the power, it will be necessary to know the work as a function of time, i.e. $A = A(t)$, so that its derivative may be found. If the velocity at which the application point of the force is displaced is known, the power may be calculated in a very simple way. From equation 2.4(4) follows:

$$\begin{aligned} dA &= \bar{F} \cdot d\bar{r} \qquad \text{from which follows} \\ P &= \frac{dA}{dt} = \bar{F} \cdot \frac{d\bar{r}}{dt} \\ \text{so that} \qquad P &= \bar{F} \cdot \bar{v} \end{aligned} \qquad 2.4(10)$$

If 1 joule of work is done in 1 second, the average power is 1 joule per second which is called **1 watt (W)**. The watt is a very small unit of power and in practice one would often use kilowatt (kW) or even megawatt (MW). The dimensions of power are $[\mathrm{P}] = [\mathrm{M\,L^2\,T^{-3}}]$.

Comment: The old British unit of power, is called the horsepower (hp). Since some old electric motors and motor-car engines are still rated in horsepower, it is helpful to remember that 1 hp ≈ 746 W ≈ 0,75 kW.

If a power of 1 W is maintained for 1 second, work of 1 J is done and it may be said that:

$$\begin{aligned} 1 \text{ watt} \times 1 \text{ second} &= 1 \text{ watt-second} = 1 \text{ joule} \\ \text{also} \qquad 1 \text{ watt-minute} &= 60 \text{ J} \\ 1 \text{ watt-hour} &= 3600 \text{ J} \\ 1 \text{ kilowatt-hour} &= 3\,600\,000 \text{ J} \end{aligned}$$

The kilowatt-hour (kWh) is not a unit of power. It is a non-SI unit of work which is of great practical importance. When SI-units were designed, it was found highly undesirable to decimalise time. The kWh is the unit used when electrical energy is sold. In 1992 1 kWh electrical energy cost about 15 c in Pretoria at normal domestic rates. As an exercise the reader can calculate that 1 kWh is required to lift a mass of 1000 kg to a height of 360 m against gravity at a constant velocity. At the mentioned rate, it is very inexpensive if the amount of work is considered.

Examples: 1. A mass of 7 kg is lifted to a height of 10 m at a constant velocity in 5 s. $g = 10\ \mathrm{m\,s^{-2}}$. Calculate the power.

This is a very simple problem. Since the velocity is constant, the instantaneous power is always equal to the average power.

Choose a frame of reference with origin at the position from which the mass is lifted with $\hat{z}$ vertically upwards. Because the mass is lifted at a constant velocity, the applied force has the same magnitude as the weight, but it is in the opposite direction.

Method 1:

$$\begin{aligned} A &= \int_P^Q \bar{F}\cdot d\bar{r} = \int_0^{10}(70\hat{z})\cdot(dx\hat{x}+dy\hat{y}+dz\hat{z}) = 700 \quad \text{J} \\ P &= \frac{dA}{dt} = \frac{\Delta A}{\Delta t} = \frac{700}{5} = 140 \quad \text{W} \end{aligned}$$

Method 2:

$$\begin{aligned} \bar{v} &= \frac{d\bar{r}}{dt} = \frac{dz}{dt}\hat{z} = \frac{\Delta z}{\Delta t}\hat{z} = \frac{10\hat{z}}{5} = 2\hat{z} \quad \text{m s}^{-1} \\ \bar{F} &= mg\hat{z} = 70\hat{z} \quad \text{N} \\ P &= \bar{F}\cdot\bar{v} = 70\hat{z}\cdot 2\hat{z} = 140 \quad \text{W} \end{aligned}$$

2. A helical spring has a force constant of 12 N m^{-1}. That means that an applied force of $\bar{F} = 12x\hat{x}$ N is required to extend the spring by x metres. Choose a frame of reference at the free end of the spring in the unextended condition with $\hat{x}$ in the direction of the extension. The extension at constant speed 0,5 m s^{-1}, begins at $t = 0$ s. Calculate (a) The work done as a function of the extension x. (b) The work as a function of the time, t. (c) The power at $t = 2$ s. (d) The average power during the first two seconds.

(a)

$$A = A(x) = \int_0^x (12\hat{x})\cdot(dx\hat{x}+dy\hat{y}+dz\hat{z}) = 6x^2 \quad \text{J}$$

(b)

$$\frac{dx}{dt} = 0,5 \quad \text{m s}^{-1} \qquad \text{so that} \qquad dx = 0,5dt \quad \text{m}$$

$$\text{and} \qquad x = \int_0^t 0,5dt = 0,5t \quad \text{m}$$

Substitute this value for x in the answer of the previous question.

$$A = A(t) = 6(0,5t)^2 = 1,5t^2 \quad \text{J}$$

(c) From the previous answer follows:

$$\begin{aligned} P &= \frac{dA}{dt} = \frac{d}{dt}(1,5t^2) = 3t \quad \text{W} \\ \text{so that} \quad P(2) &= 3\times 2 = 6 \quad \text{W} \end{aligned}$$

(d) From the second answer, the work done during the first two seconds may be calculated.

$$A(2) = 1,5\times 2^2 = 6 \quad \text{J}$$

so that the average power is given by:

$$\frac{\Delta A}{\Delta t} = 6/2 = 3 \quad \text{W}$$

3. A man uses a rope with mass per unit length of 0,5 $\text{kg}\,\text{m}^{-1}$ to hoist a bucket of water, of which the mass is 20 kg, from a well as shown in figure 2.4-4. Choose a frame of reference with origin where the man stands when the bucket is 10 m down the well and $\hat{x}$ in the direction in which the man walks. Calculate: (a) The force that the man has to exert on the rope while hoisting the bucket at a constant speed. (b) The total work required to lift the bucket to the top. (c) The power when the bucket is 2 m from the top. The man walks at a constant speed of 2 $\text{m}\,\text{s}^{-1}$. $g = 10$ $\text{m}\,\text{s}^{-2}$.

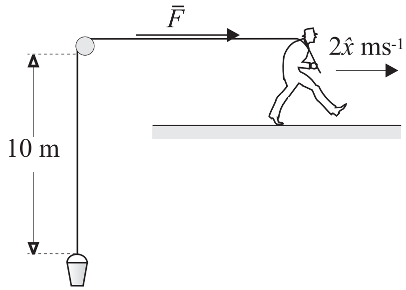


Figure 2.4-4

(a) If the bucket is hoisted at a constant speed, the force that the man exerts, is equal to the weight of the bucket and that portion of the rope which hangs down vertically. Initially the force needed to balance the weight of the bucket and 10 m of rope is equal to $(20 + 10 \times 0,5) \times 10\hat{x} = 250\hat{x}$ N. For each metre the mans walks forwards, this force decreases by the weight of 1 m of rope. The applied force is thus given by:

$$\bar{F} = \bar{F}(x) = (250 - 5x)\hat{x} \quad \text{N}$$

(b) The work done in hoisting the bucket to the top is given by:

$$A = \int_0^{10} (250 - 5x)dx = 250x - 2,5x^2|_0^{10} = 2250 \quad \text{J}$$

(c) The velocity of the man is $\bar{v} = 2\hat{x}$ $\text{m}\,\text{s}^{-1}$, and the power may be calculated as follows:

$$P = P(x) = \bar{F} \cdot \bar{v} = (250 - 5x)\hat{x} \cdot 2\hat{x} = 500 - 10x \quad \text{W}$$

When the bucket is 2 m from the top, the man has moved 8 m forwards, i.e. $x = 8$ m. The power at this position is thus given by:

$$P(8) = 500 - 10 \times 8 = 420 \quad \text{W}$$

Comment: As an extra exercise the reader may verify the following by calculation: It takes 5 seconds to hoist the bucket from the well. When the man begins to walk, the power exerted is 500 W. The average power over the 5 s needed to complete the task, is 450 W.

2.4.3 The calculation of work if the power is known

From the definition of average power, P_{av}, it is apparent that the work done in a given interval of time is equal to the average power multiplied by the time interval. In practice this might pose a problem since the average power might be valid for only the period for which it was measured. If the average power is dependent on time, it is not a very useful concept. There are, however, two applications where it is useful and is used very often:

(1) When the power is independent of time, i.e. it is constant. The average power is then the same as the instantaneous power, and it may be written that:

$$A = P_{av} \times \Delta t$$

(2) When the value of power is time dependent but **cyclic** over a relatively short period of time. If the work needs to be calculated over a period of time which is long compared to one cycle, the following will give an accurate answer:

$$A = P_{av} \times \Delta t \qquad 2.4(11)$$

The cyclic internal-combustion engine gives a typical example of a calculation of this kind. The instantaneous power of such an engine, varies greatly with time but the period of a cycle is so short that the use of the average power is a normal procedure. When a manufacturer specifies the power of an engine at a given speed, what is specified is the average power for one cycle. The power of an alternating electric current is another example where the average power during one cycle is used to calculate work as shown in equation 2.4(11).

If the power is known as a function of time, the work done during any period of time may be calculated by means of a definite integral. By definition:

$$P = \frac{dA}{dt} \qquad \text{so that} \qquad dA = Pdt \qquad \text{and}$$

$$A = A(t_2) - A(t_1) = \int_{t_1}^{t_2} Pdt \qquad 2.4(12)$$

Examples:

1. Consider problem 2 on page 90 which deals with a helical spring which is being stretched. Here the power was calculated as a function of time: $P = 3t$ W. Equation 2.4(12) may now be used to calculate the work done during the first two seconds.

$$A = \int_0^2 3t\,dt = 1,5t^2|_0^2 = 6 \quad \text{J}$$

which is in accordance with answer (d) which was calculated in a different manner.

2. Consider problem 3 on page 91. In part (c) the power was calculated as a function of the distance x: $P = P(x) = 500 - 10x$ W. Since the speed is known, it is also known that $x = 2t$ m, and the power may be calculated as a function of time as follows: $P = P(t) = 500 - 20t$ W. Once again equation 2.4(12) may be used to calculate the work done during the first 5 seconds which is the time required to hoist the bucket to the top.

$$A = \int_0^5 (500 - 20t)dt = 500t - 10t^2|_0^5 = 2250 \quad \text{J}$$

3. The use of equation 2.4(11) is illustrated in problem 1 on page 89. Here the average power during the entire interval of time, is equal to the power at every instant. The work done during the first 5 seconds is given by:

$$A = P \times \Delta t = 140 \times 5 = 700 \quad \text{J}$$

2.5 Energy

2.5.1 Work against inertia. Kinetic energy.

Consider a point mass of m kilogram which is in motion under the action of an unbalanced external force $\bar{F}$ newton. The mass accelerates along the space curve on which it moves and the force will, in general, do work. We wish to determine the effect of this work on the motion of the point mass.

From Newton's second law of motion:

$$\bar{F} = m\bar{a} = m\frac{d\bar{v}}{dt} \qquad 2.5(1)$$

On both sides calculate the scalar product with the velocity vector $\bar{v}$.

$$\bar{F} \cdot \bar{v} = m\frac{d\bar{v}}{dt} \cdot \bar{v} \qquad 2.5(2)$$

in which the left side is equal to the power, P, of the force. From the theory of vector differentiation it is known that:

$$\frac{d}{dt}(\bar{v} \cdot \bar{v}) = \bar{v} \cdot \frac{d\bar{v}}{dt} + \frac{d\bar{v}}{dt} \cdot \bar{v} = 2(\bar{v} \cdot \frac{d\bar{v}}{dt})$$

This is true because scalar products are commutative. It is known that $\bar{v} \cdot \bar{v} = v^2$ and the previous equation may be rewritten as follows:

$$\bar{v} \cdot \frac{d\bar{v}}{dt} = \frac{1}{2}\frac{d}{dt}(v^2) = \frac{d}{dt}(\frac{1}{2}v^2)$$

Using this information, equation 2.5(2) may be rewritten in the following way:

$$P = m\frac{d}{dt}(\frac{1}{2}v^2) = \frac{d}{dt}(\frac{1}{2}mv^2)$$

If the speed of the mass is v_1 at instant $t = t_1$ and v_2 at instant $t = t_2$, the work done by the external force may be calculated as follows:

$$\int_{t_1}^{t_2} P\,dt = \int_{v_1}^{v_2} \frac{d}{dt}(\frac{1}{2}mv^2)dt$$

The integral on the left side is the total work done by the force (according to equation 2.4(12)) and the calculation of the right-hand side is quite simple. The result is as follows:

$$A = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \qquad 2.5(3)$$

The result shows the following: The total work done by the unbalanced external force, $\bar{F}$, is not dependent on either the path that the particle followed, or the form of the force function. It is equal to the difference in the numerical value of the quantity $\frac{1}{2}mv^2$ at the end of the interval of time and its value at the beginning. This important quantity is called the **kinetic energy** of the particle and is indicated by the symbol E_k:

$$E_k = \frac{1}{2}mv^2 \qquad 2.5(4)$$

2.5.2 Conservative force fields and potential energy

If a force exists at each position in a given region, the **vector function of position** which describes that force, is known as a **force field**. This concept has been superficially discussed in 2.2.1.

It order to explore the relationship between forces and energy, it will be necessary to examine some important properties of certain force fields. The weight of a body, which may be taken as constant in the region for which some calculations will be made, is a simple but excellent example of such a force field.

Before the calculations are made, some very important facts have to be stressed. When a body is said to be **in equilibrium**, it is force-free in the sense that the resultant of all forces acting on it, is equal to the zero vector. If a body in a gravitational field is to be kept in equilibrium, it will be necessary to apply an external force of which the magnitude is the same as its weight but of which the direction is opposite to it.

If a body is to be displaced in a gravitational field, the magnitude and direction of the applied force will be determined by the manner in which the displacement is to be made. If the displacement is to be at a constant velocity (i.e. the kinetic energy does not change), it will have to be force-free in accordance with Newton's first law. The applied force and the weight cancel each other and the body is in equilibrium. If the body is accelerated during the displacement, a resultant force exists in the direction of the acceleration vector in accordance with Newton's second law. The force which causes the acceleration, is the resultant of the weight and the applied force.

Problem 1: Consider a frame of reference with $\hat{z}$ vertically upwards. A body with mass m kg is displaced vertically from position $\bar{r}_1 = z_1\hat{z}$ m to position $\bar{r}_2 = z_2\hat{z}$ m. Calculate the work done by (a) The applied force, $\bar{F}$, and (b) the weight, $\bar{W}$, of the body.

(a) In the chosen frame of reference the weight is given by $\bar{W} = -mg\hat{z}$ N. Since the displacement is at constant velocity, the applied force is given by $\bar{F} = mg\hat{z}$ N. The work done by the applied force, is given by:

$$A'_{12} = \int_{z_1}^{z_2} (mg\hat{z}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z}) = mgz_2 - mgz_1$$

(b) The work done by the weight, is given by:

$$A_{12} = \int_{z_1}^{z_2} (-mg\hat{z}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z}) = mgz_1 - mgz_2$$

which is equal but opposite in sign to the work done by the applied force.

Take note of the following facts: (i) If the mass is accelerated during the displacement, the work done by the applied force (which then does not have the same magnitude as the weight), will be different from that if the displacement had taken place at a constant velocity. The work done by the weight will, however, be exactly the same as that calculated in (b). (ii) If the displacement had been in the opposite direction, the calculations would have been made in the same manner but with the boundaries of the integrals exchanged, i.e. the answers would have opposite signs in each case.

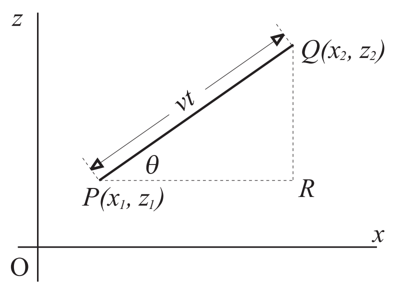


Figure 2.5-1

Problem 2: Consider the same body and frame of reference as that in the previous problem. Calculate the work done by (a) the applied force, $\bar{F}$, (b) the weight, $\bar{W}$ when the mass is displaced from position $\bar{r}_1 = x_1\hat{x} + z_1\hat{z}$ m along a straight line to position $\bar{r}_2 = x_2\hat{x} + z_2\hat{z}$ m.

(a) Moving at a constant velocity is the same as moving at a constant speed along a straight line. This requires the body to be force-free, i.e. the applied force has to be equal in magnitude but opposite in direction to the weight.

If the straight line along which the body moves makes an angle θ with $\hat{x}$ and the speed at which it moves is equal to v m s^{-1}, the velocity vector is given by:

$$\begin{aligned}
\bar{v} &= (v\cos\theta)\hat{x} + (v\sin\theta)\hat{z} \\
\text{and} \quad \bar{v} &= \frac{d\bar{r}}{dt}, \quad \text{so that} \\
d\bar{r} &= [(v\cos\theta)\hat{x} + (v\sin\theta)\hat{z}]dt \text{ and} \\
\text{and} \quad \bar{F}\cdot d\bar{r} &= (mg\bar{z})\cdot[(v\cos\theta)\hat{x} + (v\sin\theta)\hat{z}]dt = (mgv\sin\theta)dt \\
\text{and} \quad A'_{12} &= \int_0^t (mgv\sin\theta)dt = (mgv\sin\theta)t = mg(vt\sin\theta)
\end{aligned}$$

in which t is the time necessary for the body to travel from position $\bar{r}_1$ to $\bar{r}_2$ along the prescribed path. In figure 2.5-1 it may be seen that $vt\sin\theta = z_2 - z_1$, and the work is given by the following:

$$A'_{12} = mg(vt\sin\theta) = mgz_2 - mgz_1$$

which is exactly the same as the answer of problem 1(a).

(b) For the calculation of the work done by the weight, $\bar{W}$, the same procedure is followed. With $\bar{W} = -mg\hat{z}$ in the chosen frame of reference, the answer is as follows:

$$A_{12} = mgz_1 - mgz_2$$

The magnitude of this answer is the same as that of 2(a), but the sign is opposite.

Many problems may be done on the same theme. So for example the work might be calculated for both forces while the body is moved from P to R along a straight line at a constant speed and then from R to Q in a similar manner. The work for both forces along the same route but in the opposite sense, is another possibility. From all these answers the following conclusions may be drawn:

(i) The work done by the weight vector, is independent of the way in which and also the path along which the body is moved. For a given mass, the answer depends upon the difference between the vertical heights of the initial and final positions only. The work done by the applied force will only be given by the calculated answers if the body is moved at a constant speed and when no frictional forces are present.

(ii) If the body is taken along a closed curve (i.e. the initial and final positions coincide), the work done by the weight will be equal to zero. This follows from the fact that the work done, has the same magnitude but the opposite sign when the body is moved between two positions in reversed sequences. The work done by the applied force will, in general, not be zero. If the body accelerates or if friction is present, the work done might not be equal to zero.

(iii) The work done by the weight, is equal to the difference in the quantity mgz at the beginning of the path and that at the end (*note the order in which the quantities are mentioned*), irrespective of the path which is followed. The important quantity, E_p, is defined as the **gravitational potential energy** of the body.

$$E_p = mgz \tag{2.5(5)}$$

in which z is the *vertical component* of the position vector. This formula is only approximately correct in a limited region (we assumed that the weight of the body was constant) and the correct expression will be derived later.

The value of E_p surely depends upon the frame of reference in which it is measured. Each observer has the right to choose his own frame of reference and it would seem that each would ascribe a different numerical value to it. This is correct, but all problems about systems near the surface of the earth are concerned about the *difference* in gravitational potential energy only. All observers will find the same value for this, irrespective of their frames of reference.

The facts which the calculations revealed about a gravitational field, are characteristic of a number of forces in nature. When a force field exhibits these properties, it is known as a **conservative force field**. A full description of the properties of such fields, requires a more extensive knowledge of vector analysis. With the knowledge at our disposal, we may summarise the properties of conservative force fields as follows:

1. The work done by a conservative force field does not depend upon the path between the initial and final points. It does, however, depend upon the co-ordinates of these two positions.

2. The work done by a conservative force along a closed path is always identically equal to zero. It may be written for a conservative force field, $\bar{F}$, that:

$$\oint \bar{F} \cdot d\bar{r} \equiv 0 \tag{2.5(6)}$$

3. The work done by a conservative force field, is equal to the potential energy at the initial position minus that at the final position. (*Comment: Take note of the order.*) If $\bar{F}$ is a conservative force field, then:

$$\int_1^2 \bar{F} \cdot d\bar{r} = (E_p)_1 - (E_p)_2 \tag{2.5(7)}$$

The limits of the integral refer symbolically to the initial and final positions. If these two limits are interchanged, the signs of the two terms on the right-hand side will change. It is of prime importance to take note of the order and signs of the integral and the potential energies.

Equation 2.5(7) gives the relationship between a conservative force field and the potential energy associated with it, in the form of an integral. This may also be written in the form of a differential equation. For the simple case of a constant gravity field, the differential equation is as follows:

$$\text{weight} = \bar{W} = -\frac{d}{dz}(E_p)\hat{z} = -\frac{d}{dz}(mgz)\hat{z} = -mg\hat{z}$$

A similar result may be written for every other conservative force field. It means that for every conservative field there exists a scalar function which is called the **potential energy function** which has the property that the force field is given by its negative gradient. The negative sign describes the fact that the potential energy decreases in the direction of the force vector. For a conservative force field, $\bar{F}$, and its corresponding potential energy function the following is valid:

$$\bar{F} = -grad(E_p) \qquad 2.5(8)$$

The gradient which is represented symbolically by "*grad*" in 2.5(8), is actually a vector operator that will be understood fully when the reader is skilled in partial differentiation and vector analysis. In Cartesian co-ordinates it is indicated by the symbol ∇ (pronounced **nabla**) which has the following meaning:

$$\begin{aligned} \nabla &\equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \\ \text{so that} \qquad \bar{F} &= -\nabla(E_p) = -\left(\frac{\partial E_p}{\partial x}\hat{x} + \frac{\partial E_p}{\partial y}\hat{y} + \frac{\partial E_p}{\partial z}\hat{z}\right) \end{aligned}$$

In the case of a force field with exhibits spherical symmetry (i.e. a radial field), this relationship may be written as follows:

$$\bar{F} = -\frac{\partial E_p}{\partial r}\hat{r} = -\frac{dE_p}{dr}\hat{r}$$

Because $\bar{F}$ and E_p depend on r only, the difference between partial and ordinary differentiation ceases to exist. Partial differentiation is explained in the introduction to chapter 5.

2.5.3 The principle of energy conservation

The content of this principle is illustrated for the special conditions which exist in the following two problems.

Problem 1: A mass of m kg is projected vertically upwards at an initial speed of v_0 m s^{-1}. Calculate (a) the maximum height to which it will rise, (b) the sum of the kinetic and gravitational potential energies at the projection point and also at maximum height.

Choose a frame of reference with origin at the initial position with $\hat{z}$ vertically upwards and commence the measurement of time when the mass leaves the origin. In this system $\bar{a} = -g\hat{z}$ m s^{-2}, $\bar{v}(0) = v_0\hat{z}$ m s^{-1} and $\bar{r}(0) = \bar{0}$. Let h be the maximum height.

(a)
$$\begin{aligned} \int_0^h -g\,dz &= \int_{v_0}^0 v_z\,dv_z \\ -g\,z|_0^h &= \frac{1}{2}v_z^2|_{v_0}^0 \qquad \text{from which follows} \\ h &= v_0^2/2g \end{aligned} \qquad 2.5(9)$$

(b) In the chosen frame of reference, the gravitational potential energy of the mass is zero at the origin and its kinetic energy $\frac{1}{2}mv_0^2$. At maximum height the mass is at rest and thus has zero kinetic energy and gravitational potential energy of mgh.

$$(E_p + E_k)_{\text{at origin}} = 0 + \frac{1}{2}mv_0^2 \qquad 2.5(10)$$
$$(E_p + E_k)_{\text{at top}} = mgh + 0 \qquad 2.5(11)$$

Substitute the value of h which was calculated in 2.5(9) into 2.5(11). From this follows:

$$mgh = mg(v_0^2/2g) = \frac{1}{2}mv_0^2$$

which shows that the energies in equations 2.5(10) and 2.5(11) are the same.

Problem 2: First calculate $v = v(z)$ for the previous problem and then the sum of the kinetic and gravitational potential energies at a point before it reaches maximum height.

$$\begin{aligned} \int_0^z -g\,dz &= \int_{v_0}^v v_z\,dv_z \\ -gz &= \frac{1}{2}v^2 - \frac{1}{2}v_0^2 \end{aligned}$$

Multiply this equation by the mass, m, of the body.

$$\begin{aligned} -mgz &= \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \\ \text{so that} \qquad \frac{1}{2}mv_0^2 &= mgz + \frac{1}{2}mv^2 \end{aligned}$$

which is equal to the answers of problem 1. The following is thus true:

$$(E_p + E_k)_{\text{at origin}} = (E_p + E_k)_{\text{at top}} = (E_p + E_k)_{\text{any position}}$$

Result: *Under the action of gravity (a conservative force) the sum of the kinetic energy and the gravitational potential energy remains constant. Some prefer to call the sum of these two energies, the* **mechanical energy** *of the system.*

In the same manner it may be shown that the same result is valid for any other conservative force field. This result is a special case of a general phenomenon known as the **principle of energy conservation** which is also known as the **law of energy conservation**. This principle is a powerful tool in the solution of problems in kinematics and dynamics. When it is used, it may not be necessary to know the forces which act, but a prerequisite is the knowledge that they are conservative.

Before examples of problems are presented, it will be necessary to calculate the potential energy associated with a few other conservative forces.

2.5.4 The potential energy of a gravitational field

In the previous section the function $E_p = mgz$ was used for the potential energy of a mass of m kilogram of which the vertical component of its position is z metre. This function is not correct because g is not a constant (see equation 2.2(5)), a fact which was disregarded. It is perfectly in order to use the function $E_p = mgz$ in calculations if the differences in height are much less than the distance between the mass and the centre of mass of the earth. On the other hand, it would be totally incorrect to use it in a problem about a rocket launched to the moon or a satellite in an orbit around the earth.

The mass of the earth is M kilogram. Choose a frame of reference with its origin at the earth's centre of mass. A mass of m kilogram is taken from position $\bar{r}_1$ to position $\bar{r}_2$ in the earth's gravitational field along the path shown in figure 2.5-2. We wish to calculate the change in gravitational potential energy of the system. The force acting on the mass of m kilogram (its weight) is given by Newton's gravitational law. In the chosen frame of reference, it is as follows:

$$\bar{F} = -\frac{GMm}{r^2}\hat{r}$$

A radial field is one which points towards or away from a point which is usually chosen to be the origin. The weight of the mass m is also a radial field which is directed towards the centre of mass of the earth and may thus be written in the form $\bar{F} = -|\bar{F}|\hat{r}$. In the calculation which follows, the scalar product $\bar{F} \cdot d\bar{r}$

is used. From figure 2.5-2, it can be seen that

$$\bar{F} \cdot d\bar{r} = -|\bar{F}|\hat{r} \cdot d\bar{r} = -|\bar{F}|dr \qquad \text{in which} \qquad dr = \hat{r} \cdot d\bar{r} \qquad 2.5(12)$$

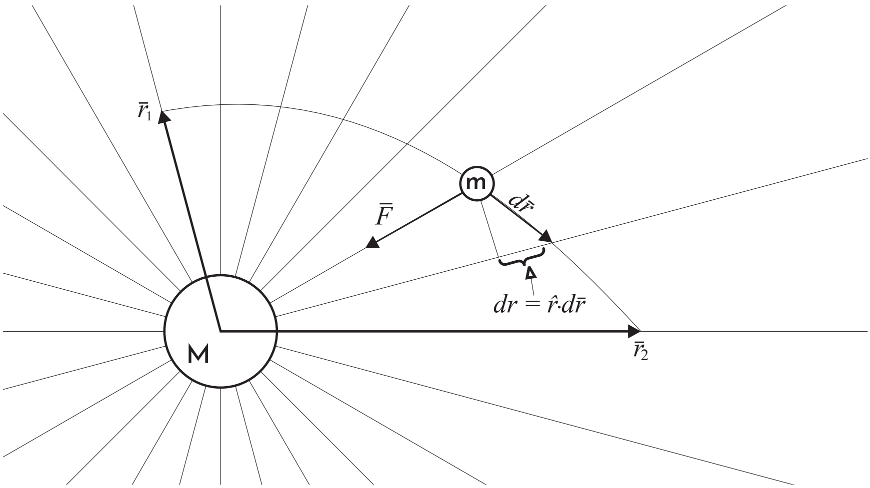


Figure 2.5-2

Comment: It is important to understand the difference between $|d\bar{r}|$ and $dr = d|\bar{r}|$. The first is an infinitesimal distance along the space curve on which the mass moves and the second an infinitesimal change in the length of the position vector of the mass.

It should be clear that the important result of 2.5(12) is valid for all radial fields and it is always used in the calculation of work and energy in such fields.

The change in gravitational potential energy of the system is given by application of 2.5(7) as follows:

$$\begin{aligned} (E_p)_1 - (E_p)_2 &= \int_{r_1}^{r_2} (-GMm/r^2)\hat{r} \cdot d\bar{r} = \int_{r_1}^{r_2} (-GMm/r^2)dr \\ &= GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad 2.5(13) \end{aligned}$$

The answer depends only upon the radial distances r_1 and r_2, and not on the path which was followed.

In many problems it is convenient to choose the reference position where the potential energy is zero, at $r = \infty$. If the substitution $r_2 = \infty$ is made in equation 2.5(13), the gravitational potential energy may be found for any position,

$\bar{r}$ as follows:

$$E_p = -\frac{GMm}{r} \qquad 2.5(14)$$

This choice of reference position for the potential energy has the effect that the function consists of a single term. Any other position, with the exception of $r = 0$ (*why?*), would have worked as well but then the potential energy function would be more complicated.

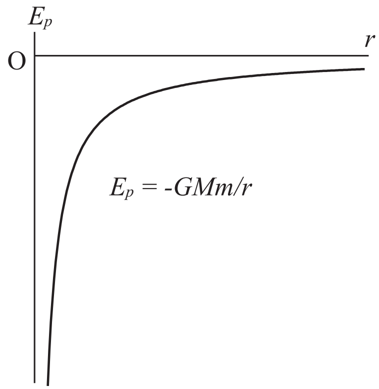


Figure 2.5-3

A graphic representation of the potential energy function is shown in figure 2.5-3. It is known as an **attraction potential** or a **potential well**.

A result of the choice of the reference position is that the function has a maximum value of zero. If this function is used and the sum of the potential and kinetic energies of a system of two bodies is negative, the bodies are said to be **bound** and if it is positive, they will be **unbound** or **free**. If a spacecraft is launched from Earth, it will always return if the sum of its potential and kinetic energies is negative. t If the sum is zero it will have just enough energy to escape from Earth's gravitational field. For zero and positive values, it will never return.

Using this function, it may be shown that equation 2.5(8) as amended for a field with spherical symmetry, gives the correct function for the weight of a body:

$$\bar{W} = -\frac{d}{dr}(E_p)\hat{r} = -\frac{d}{dr}(-GMm/r)\hat{r} = -(GMm/r^2)\hat{r}$$

2.5.5 Electrical potential energy

Consider an electric point charge q at position $\bar{r}$ relative to another point charge of Q coulomb. The force exerted on q is given by Coulomb's law:

$$\bar{F} = \frac{kQq}{r^2}\hat{r}$$

As in the case of the gravitational field, it is useful to take the electrical potential energy as zero where r is infinitely large. With this reference position the electrical potential energy of the system becomes:

$$E_p(r) = \int_r^\infty (kQq/r^2)\hat{r} \cdot d\bar{r} = \int_r^\infty (kQq/r^2)dr = \frac{kQq}{r} \qquad 2.5(15)$$

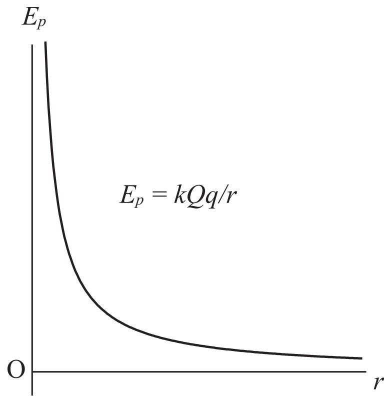


Figure 2.5-4

Contrary to the gravitational potential energy which is always negative, electrical potential energy may be either positive or negative. If the two charges have opposite signs, it represents an attractive potential or potential well with the same shape as that in figure 2.5-3. In such a case the charges may be either bound or unbound just as for two masses. If the two charges have the same sign, the function represents a **repulsive potential** or a **potential barrier** of which an illustration is shown in figure 2.5-4. In such a case the charges can in no way be bound because they always repel each other.

In the study of electrical phenomena, it is preferable not to work with the total potential energy of a system but rather with the potential energy per unit positive charge at a given position which is called the **electric potential** at that position. It is usually indicated by the symbol V. From equation 2.5(15) the potential due to a point charge Q, may be calculated by substituting $+1$ for the charge q as follows:

$$V = V(r) = E_p/q = kQ/r \qquad 2.5(16)$$

This function is valid only for the electric potential due to a *point charge*. For other **charge distributions**, the potential functions will, in general, be different.

The unit of electric potential is **joule per coulomb** which is also called **volt (V)**. If the electric potential, V, at a given position is known, the potential energy of a point charge, q, placed at that position, may be calculated as follows:

$$E_p = qV \qquad 2.5(17)$$

The difference in electric potential at two positions in an electric field, is known as the **electric potential difference**. In terms of the electric field strength which was defined in 2.2.2, it is given by the following:

$$V_{ab} = V_a - V_b = \int_a^b \bar{E} \cdot d\bar{r} \qquad 2.5(18)$$

This expression follows directly from the important equation 2.5(7) and the definition of an electric field. *Comment: Take note of the signs and the order in which a and b occur.*

As a supplementary exercise the reader may illustrate that the force which acts on one of a pair of electric point charges, is given by the negative gradient of the potential energy function, 2.2(15), and also that the electric intensity is given by the negative gradient of the potential function, 2.2(16). These results are of prime importance in the study of electrical phenomena.

2.5.6 Elastic potential energy

Material objects are changed when they are subjected to forces. The volume of a gas at constant temperature will decrease if the pressure is increased, a length of metal wire will increase if a tensile force is applied and a piece of Plasticine will change its shape when pressure is exerted on it. All materials do not exhibit the same behaviour when changing shape. If the force which elongates a length of rubber band is removed, it usually regains its original length. In this case the rubber band had an **elastic** deformation. Plasticine will not restore to any degree if the deforming force is removed and its deformation is said to be **plastic**. In some plastic deformations, partial restoration is possible and it is said that the **elastic limit** has been exceeded. This will be the case if a piece of metal wire is stretched too much or a helical spring extended beyond this limit.

During plastic deformations the material is always damaged and the work done during such a deformation will result in a rise in temperature of the material.

If a deformation is perfectly elastic, the internal forces in the material are conservative and the work done during such a deformation, is converted to **elastic potential energy**. If no deformation exists, the elastic potential energy is zero. The elastic deformation of a helical spring will be studied to illustrate such a process.

Within its elastic limit, the magnitude of the elastic force (restoring force), F, in a stretched spring is directly proportional to the elongation, x. Thus $F = kx$ in which the proportionality constant, k, is known as the **force constant** or **stiffness coefficient** of the spring. The relationship between the force and the elongation is known as **Hooke's law**. A graphic representation is shown in figure 2.5-5(a). The change in potential energy is given by the important equation 2.5(7) which is valid for all conservative force fields.

Consider a helical spring of which the upper point is firmly clamped. It will be assumed that the spring is so light that its weight will cause no measurable elongation. Choose a frame of reference with origin at the lower end of the spring and $\hat{x}$ vertically downwards, the direction in which the extension will take place. The elastic force (or restoring force) in the spring is given by $\bar{F} = -kx\hat{x}$. An elongation from position $\bar{r}_1 = \bar{0}$ m to position $\bar{r}_2 = x\hat{x}$ m causes a change in the elastic potential energy given by:

$$\begin{aligned}(E_p)_1 - (E_p)_2 &= \int_0^x (-kx\hat{x})\cdot(dx\hat{x}+dy\hat{y}+dz\hat{z}) = \int_0^x -kx\,dx \\ &= -\frac{1}{2}kx^2|_0^x = -\frac{1}{2}kx^2\end{aligned}$$

But $(E_p)_1 = 0$ since the spring is then unstretched. The elastic potential energy of a stretched spring is therefore given by:

$$E_p = \frac{1}{2}kx^2 \qquad 2.5(19)$$

This amount of energy is represented by the shaded area in figure 2.5-5(a).

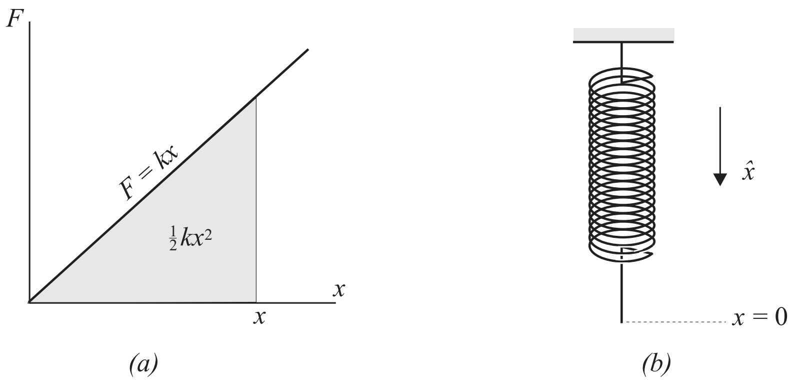


Figure 2.5-5

The elastic force or restoring force in the spring, is given by the gradient of the potential energy function:

$$\bar{F} = -\frac{d}{dx}(\frac{1}{2}kx^2)\hat{x} = -kx\hat{x}$$

2.5.7 Work done against non-conservative forces

A feature of all changes in systems in which only conservative forces act, is that they are reversible. Some forces in nature are different and they do not qualify as conservative. The reason for this will only become apparent when the reader has made a study of thermodynamics (or statistical mechanics).

If a body is dragged over a rough surface from position A to position B, the temperature rises as its **internal energy** is increased. Otherwise than with conservative forces, the work done against friction, will certainly depend on the path followed from the initial to the final position. The body, of which the temperature is now higher, will not start cooling spontaneously and retrace its path to the starting point. In this case mechanical energy is not conserved. If other forms of energy are considered, a more general principle of energy conservation may be formulated.

Consider the case where two billiard balls collide. The material from which they are made is highly elastic, but the collision produces audible sound which means that mechanical energy is removed from the system. A similar interaction may also happen on an atomic scale where two atoms interact (collide) with the emission of a photon. Such interactions and also the example of a body being dragged over a rough surface are **irreversible** and are called **dissipative** processes. Even in these cases the total amount of energy remains unchanged if all the relevant forms of energy are taken into account.

The principle of energy conservation may be formulated as follows: *Energy cannot be created or destroyed; only conversion from the one kind to another is possible.* As a variation on the theme, one might say that *the energy of an isolated system remains unchanged, no matter what happens within the system.* When, however, dissipative forces act, the conversions will be irreversible.

In supplying the energy needs of modern society, one can think of many energy "sources" such as internal-combustion engines, electric batteries, thermocouples, nuclear plants etc. These so-called sources are all energy converters by means of which one form of energy is transformed to another. In each converter a portion of the energy is lost through dissipative processes. This causes the converter to be less than 100% efficient. The **efficiency**, η of an energy converter is defined as the ratio between the energy (or power) which it makes available and the energy (or power) that is supplied to it from another primary source.

$$\eta = \frac{\text{output energy}}{\text{input energy}} = \frac{\text{output power}}{\text{input power}} \qquad 2.5(20)$$

It is of importance that the reader realises that classification of energy as mechanical (kinetic and potential) and non-mechanical (electrical, thermal, chemical etc.) is very artificial. What is called heat is in fact internal energy transformed from one system to another due to differences in temperature. The internal energy may be stored as the kinetic energy of atoms and molecules or even binding energy which involves electric forces. So-called chemical energy is stored as the electrical potential energy of atoms which are bound to other atoms. The relativity theory of Einstein reveals that even mass is a form of energy and this is expressed by his famous equation:

$$E = mc^2 \qquad 2.5(21)$$

in which E joules of energy are equivalent to m kilogram mass and $c =$ the speed of light in free space. In nuclear reactors and atomic bombs energy is made available by the conversion of mass to energy according to this equation. In the process of electron-positron pair formation, radiant energy is used to create mass and the quantities are given by 2.5(21).

It is an interesting exercise to investigate so-called chains of energy conversions. The radiant energy from the sun is powered by nuclear fusion in which hydrogen is converted to helium. Plants can utilise this energy in the process of photosynthesis after which it is stored as chemical energy. Combustion of plant material supplies energy for the production of steam under high pressure (potential energy), which can drive a turbine (kinetic energy). The turbine may be used to drive a dynamo and that supplies electrical potential energy to electrons. When these electrons flow through an electrical conductor with resistance, its temperature will rise. It is interesting to know that one's dinner is cooked by sunshine which reached the earth many millions of years ago to become trapped in coal!

In the examples which follow, only problems will be considered in which conservative forces act. When the reader wishes to use the principle of energy conservation for the solution of a problem, it will be necessary to account for *all* the energy. If the forces which act are conservative, the task is usually simple.

2.5.8 Examples of problems in which energy conservation is used

1.

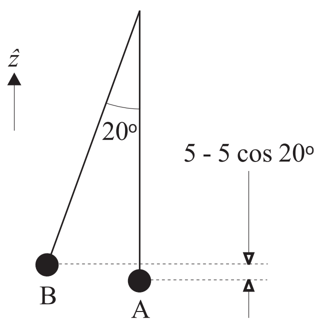


Figure 2.5-6

A point mass of m kilogram is suspended from a 5 metre long light cord to form a simple pendulum. It is set in motion with an amplitude of 20°. (The amplitude is the maximum angular deviation from the equilibrium position.) Calculate the speed when it passes through the lowest point. $g = 10$ m s^{-2}. See figure 2.5-6.

Choose a frame of reference with origin at the equilibrium position (A) and $\hat{z}$ vertically upwards. Let the position of maximum deviation be B.

Since the system is in a conservative force field (gravity):

$$(E_p + E_k)_A \quad = \quad (E_p + E_k)_B$$

$$mgz_A + \frac{1}{2}mv_A^2 = mgz_B + \frac{1}{2}mv_B^2$$

But since $z_A = 0$ and $v_B = 0$ it follows that:

$$v_A^2 = 2gz_B$$
$$= 2 \times 10 \times (5 - 5\cos 20^\circ)$$

from which $v_A = 2,456 \quad \mathrm{m\,s^{-1}}$

2.

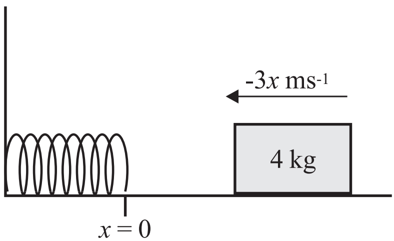


Figure 2.5-7

Figure 2.5-7 shows a mass of 4 kg sliding at a constant velocity of -$3\hat{x}$ $\mathrm{m\,s^{-1}}$ on a smooth surface of which the friction may be disregarded. At $x = 0$ it strikes the free end of an undeformed elastic helical spring of which the force constant is 144 $\mathrm{N\,m^{-1}}$. Calculate the compression of the spring when the mass comes to rest. Assume that the deformation of the spring occurs within its elastic limit and that the windings do not touch each other when the mass is at rest.

The only force which influences the motion of the mass is the elastic force in the spring which is conservative. Therefore:

$$(E_k \text{ of mass} + \text{elastic } E_p \text{ of spring}) = \text{constant}$$
$$\frac{1}{2} \times 4 \times 3^2 + 0 = 0 + \frac{1}{2} \times 144 \times x^2$$

from which follows $x = 0,5$ m

3. A light elastic helical spring with force constant k $\mathrm{N\dot{m}^{-1}}$ is suspended from the top next to a vertical metre rule on which its extension is measured. In the unextended condition the pointer is at position 1 (See figure 2.5-8). When a mass of m kilogram hangs in equilibrium from the lower end, the pointer is at position 3. The extension of the spring in this position is r metre. The mass is then lifted A metre ($A < r$) vertically to position 2 and then allowed to oscillate between positions 2 and 5. Position 5 is B metre under the equilibrium position (3).

(a) Calculate the energy of the system in position 2. Choose position 3 as the zero-point for the gravitational potential energy.

(b) Show that $A = B$.

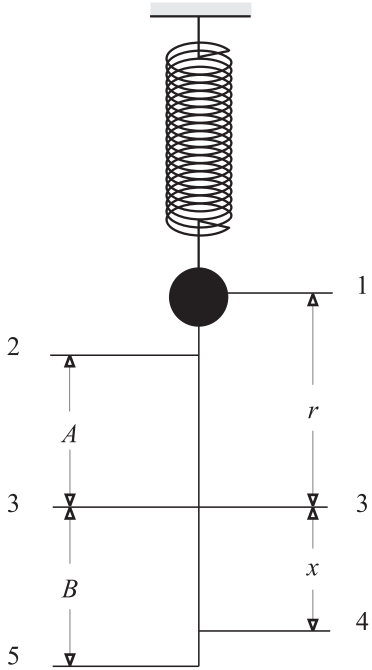


Figure 2.5-8

(c) Position 4 is x metre below the equilibrium position, 3. Calculate the velocity of the mass in this position in terms of x. Disregard friction.

Two conservative forces are involved in this problem viz. gravity and the elastic force in the spring. Since the spring is light, its mass may be disregarded, and only three forms of energy need to be taken into account. They are as follows: the kinetic energy of the mass, the gravitational potential energy of the mass and the elastic potential energy of the spring.

In equilibrium position 3, the extension of the spring is r metre which is caused by the weight of mass m and therefore $mg = kr$. In the calculations which follow mg may be replaced by kr and vice versa. This will allow the simplification of some expressions in these calculations.

(a) At position 2 the mass is at rest. The total energy at this position is given by:

$$\begin{aligned}
(E_T)_2 &= (E_k \text{ of mass})_2 + (E_p \text{ of mass})_2 + (E_p \text{ of spring})_2 \\
&= 0 + mgA + \frac{1}{2}k(r-A)^2 \\
&= 0 + mgA + \frac{1}{2}kr^2 - krA + \frac{1}{2}kA^2 \\
&= \frac{1}{2}kr^2 + \frac{1}{2}kA^2 \quad \text{because } mg = kr \text{ so that } mgA - krA = 0
\end{aligned}$$

(b) At position 5 the mass is again at rest and the total energy is calculated in the same manner as that for position 2.

$$\begin{aligned}
(E_T)_5 &= (E_k \text{ of mass})_5 + (E_p \text{ of mass})_5 + (E_p \text{ of spring})_5 \\
&= 0 + mg(-B) + \frac{1}{2}k(r+B)^2 \\
&= 0 - mgB + \frac{1}{2}kr^2 + krB + \frac{1}{2}kB^2 \\
&= \frac{1}{2}kr^2 + \frac{1}{2}kB^2 \quad \text{because} \quad krB - mgB = 0
\end{aligned}$$

The forces involved are conservative and therefore the total energy at position 5 must be equal to the total energy at position 2. This can only be the case if $A = B$. The two turning points are thus symmetrical about the equilibrium position.

(c) At position 4 the mass is not at rest and the total energy is given by:

$$\begin{aligned}
(E_T)_4 &= (E_k \text{ of mass})_4 + (E_p \text{ of mass})_4 + (E_p \text{ of spring})_4 \\
&= \frac{1}{2}mv^2 + mg(-x) + \frac{1}{2}k(r+x)^2 \\
&= \frac{1}{2}mv^2 - mgx + \frac{1}{2}kr^2 + krx + \frac{1}{2}kx^2 \\
&= \frac{1}{2}mv^2 + \frac{1}{2}kr^2 + \frac{1}{2}kx^2 \quad \text{because} \quad krx - mgx = 0
\end{aligned}$$

According to the principle of energy conservation, the total energy at position 4 must be equal to that at the other positions. The energy at 4 is equated to that which has been calculated for position 2.

$$\begin{aligned}
\frac{1}{2}mv^2 + \frac{1}{2}kr^2 + \frac{1}{2}kx^2 &= \frac{1}{2}kr^2 + \frac{1}{2}kA^2 \\
\text{from which follows} \qquad v &= \pm\left(\frac{k}{m}(A^2 - x^2)\right)^{\frac{1}{2}}
\end{aligned}$$

The two values of v differ in sign only and correspond to the upward and downward motions of the mass. The function contains the square of x and is therefore independent of its sign. This shows that the functions are also valid for values of x above the equilibrium position. The reader may verify this by repeating the calculation of the energy at position 4 when it is chosen to be *above* position 3. This result may also be derived by using equation 1.6(2).

4. A proton of mass $1,673 \times 10^{-27}$ kg and electric charge $1,602 \times 10^{-19}$ C begins at rest in the ion source of a Van de Graaff accelerator. The ion source is at an electric potential of 2×10^6 V. Calculate the speed of the proton when it leaves the accelerating field. The electric potential is zero outside the accelerating field.

The electric field is conservative and the total energy is conserved. Therefore:

$$\begin{aligned}
(E_k + \text{electric } E_p)_{\text{in ion source}} &= (E_k + \text{electric } E_p)_{\text{outside accelerating field}} \\
0 + (1,602 \times 10^{-19}) \times (2 \times 10^6) &= \frac{1}{2}(1,673 \times 10^{-27})v^2 + 0 \\
\text{from which follows} \qquad v &= 1,957 \times 10^7 \quad \mathrm{m\,s^{-1}}
\end{aligned}$$

2.5.9 Potential energy and equilibrium

The motion of a system of mass points may be limited by the application of **constraining forces**. This causes each system to have a limited number of **degrees of freedom**. A wheel that rotates around a fixed axis, has only one degree of freedom. Knowledge of the angle of rotation at each instant for such a wheel provides all the information that might be required about its motion. Such a co-ordinate is called a **generalised co-ordinate**. A mass moving on a horizontal plane at the one end of a cord of unspecified length with its other end fixed to the plane, has two degrees of freedom and its generalised co-ordinates may be chosen to be the length of the cord (r) and its angle of rotation (θ).

If a point mass is constrained to motion under the action of a conservative force, the potential energy associated with that force may be written as a function of a suitable generalised co-ordinate. A fairly simple relationship exists between the properties of the potential energy function and the positions in which the mass will be in equilibrium. This relationship may be derived directly from the **principle of virtual work** which is an elegant method to calculate unknown forces which act in a system which is in equilibrium. This principle will not be treated here. Only the result of interest is given.

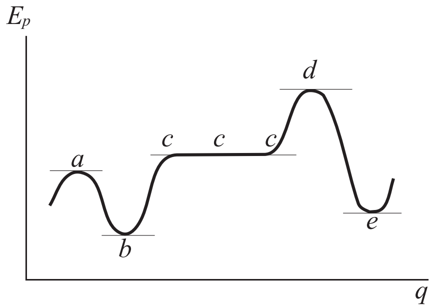


Figure 2.5-9

Consider a body of which the potential energy depends on a single generalised co-ordinate, q. Then

$$E_p = E_p(q)$$

A graphic representation of a hypothetical potential-energy function is given in figure 2.5-9. It will be found that the system is in equilibrium at each value of q for which the energy is a maximum or a minimum and also where the energy remains constant with changing q, i.e. at each value of q for which $dE_p/dq = 0$. Where the potential energy is a minimum, the equilibrium is **stable**. That means that the system will revert to its initial configuration after a relatively small displacement; it will always return to the equilibrium position. Stable equilibrium is represented by points b and e on the graph.

At values of q where the potential energy function has a maximum, the equilibrium is **unstable**. That implies that the system cannot restore itself after a slight displacement and will not, of its own accord, return to the equilibrium position. Points a and d on the graph respresent unstable equilibrium states. In regions where the potential energy remains constant with changing

q, the equilibrium is **neutral**. Neutral equilibrium has the property that it is not influenced by a local disturbance. Points c on the graph represent neutral equilibrium.

To determine the equilibrium positions of a system and the kind of equilibrium, the mathematical theory for determining local maximum and minimum values of a function is used. It will require the use of at least second-order derivatives to determine the nature of these stationary points if they exist.

Figure 2.5-10 shows three bodies which can each exhibit all three kinds of equilibrium. Take note of the constraining forces. The sphere and the cone are forced by their weight to keep contact with the surfaces on which they rest and the motion of the bar is limited by a fixed pivot.

The wonderful balancing acts performed by circus artistes, illustrate that it is possible to find a system in unstable equilibrium but this condition is extremely rare in nature. One could thus say that nature gives preference to stable equilibrium except where the constraining forces allow only neutral equilibrium to exist. In a different way it could be said that *the potential energy of each system tends to be a minimum.* In practice it means that the force field which is responsible for the potential energy, also tends to keep it as small as possible.

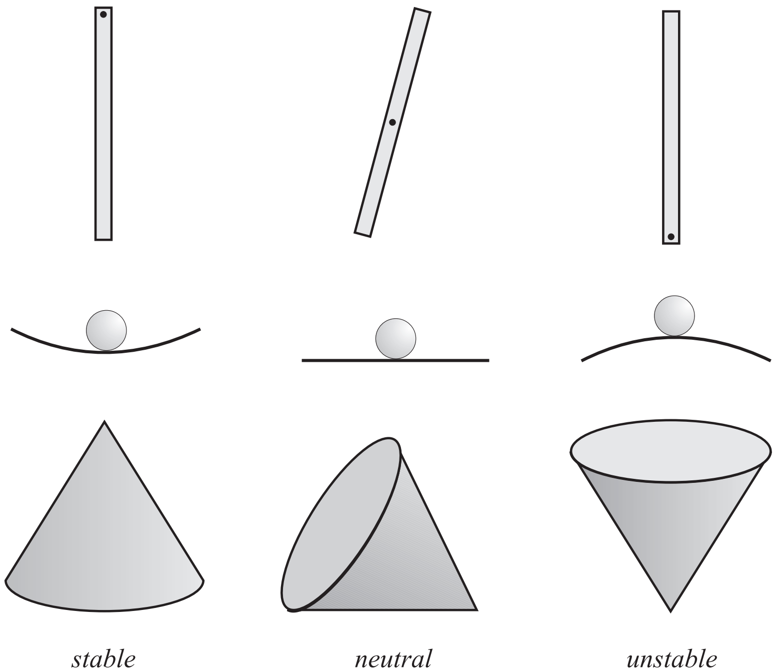


Figure 2.5-10

Example:

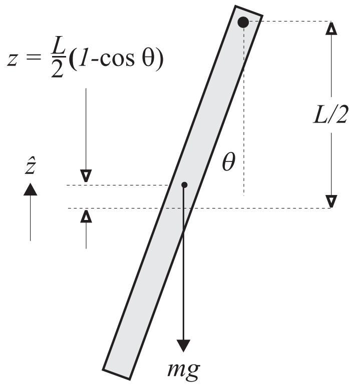


Figure 2.5-11

A thin uniform homogeneous rod has a mass of m kilogram and a length of L metre. Its centre of mass coincides with its geometric centre. It is pivoted at one end and hangs vertically downwards. It is then given an angular displacement of θ. (a) Take its gravitational potential energy as zero when $\theta = 0$ and calculate $E_p = E_p(\theta)$. (b) Calculate the values of θ for which it will be in equilibrium. Investigate the nature of the equilibrium in each case. See figure 2.5-11.

(a) If the rod is displaced by angle θ, its centre of mass is lifted vertically by a distance z given by:

$$z = \frac{L}{2} - \frac{L}{2}\cos\theta$$

The potential energy of the rod is given by:

$$E_p = E_p(\theta) = \frac{mgL}{2}(1 - \cos\theta)$$

(b) $dE_p/d\theta = (mgL/2)\sin\theta$ which is equal to zero when $\theta = 0$ and $\theta = 180°$. At these two positions the rod will experience equilibrium.

$d^2E_p/d\theta^2 = (mgL/2)\cos\theta$ which is positive when $\theta = 0$. At this position the potential energy is a minimum and the equilibrium is stable. When $\theta = 180°$, the second-order derivative is negative, indicating a maximum and thus corresponding to unstable equilibrium.

2.6 Collisions and impulse

2.6.1 Different kinds of collision

At this stage the reader should know enough dynamics to be able to solve most problems concerning collisions. Collisions play a most important role in the study of atomic and nuclear physics. For this reason a section will be devoted to collisions as a special application of the principles of conservation of momentum and energy. During a collision the kinetic energy of a system may remain the same, diminish or even increase. Each possibility will be discussed and illustrated with examples.

When two protons move towards each other, the coulomb repulsive force increases rapidly with decreasing distance between them and it may become so large that they might never come into contact. The interaction which takes place will, however, change their velocities drastically. Although the protons are never in contact, the interaction qualifies for what is usually described as a collision. The force between the protons is conservative and the total kinetic energy at the onset of the interaction will be equal to that after the collision. A collision of this kind is called an **elastic collision** or **interaction**.

When two billiard balls or hard steel balls collide, the collision will be approximately elastic. Contact during such a collision usually lasts only a few milliseconds, the bodies exert enormous forces on each other and the deformation can be considerable.[4] The deformations are nearly perfectly restored by the elastic forces in the bodies, but some kinetic energy may be lost where the elastic limits of the materials are exceeded. This causes material damage and a rise in temperature. Kinetic energy may also be removed from the system by the emission of sound waves. An interaction in which the kinetic energy of the system diminishes, is called an **endoergic collision** or **endoergic interaction**. An endoergic interaction may occur in an atomic level when some of the kinetic energy is lost to electronic excitation of one or both of the particles involved.

An extreme case of an endoergic process occurs when no restoring forces are present. A ball of soft Plasticine which falls on the floor is an example of such a collision. It is described as **completely inelastic**. In a completely inelastic collision, no mechanism exists to separate the two bodies and they will have to move together after the interaction. When a perfectly inelastic interaction is studied in a frame of reference in which the resultant momentum is zero (i.e. the so-called **centre of mass system**), all the kinetic energy will be converted to other forms of energy. In a different frame of reference, the change in kinetic energy may be calculated quite simply as will be shown in the examples.

If point mass m_1 and velocity $\bar{v}_1$ interacts with point mass m_2 of which the velocity is $\bar{u}_1$, their relative speed is initially $|\bar{u}_1 - \bar{v}_1|$. If their velocities are $\bar{v}_2$ and $\bar{u}_2$ after the collision, their relative speed will be $|\bar{u}_2 - \bar{v}_2|$. A useful parameter (designed and used by Isaac Newton) which is used to describe inelastic collisions, is the **coefficient of restitution** or the **collision coefficient**, e, which is defined as follows:

$$e = \frac{|\bar{u}_2 - \bar{v}_2|}{|\bar{u}_1 - \bar{v}_1|}$$

[4]See excellent high-speed photographs by Stephen Dalton in his book *Split Second*

From this definition it should be clear that $e = 0$ for a perfectly inelastic collision and $e = 1$ for one which is perfectly elastic. For collisions in which only partial restitution takes place, $0 \leq e \leq 1$.

Interactions may occur during which some form of energy is converted into kinetic energy. An aircraft in flight may be struck by a warhead with the ensuing release of a large amount of chemical energy which is converted to kinetic energy. Such an interaction is known as an **exoergic collision** or **exoergic interaction**. Exoergic interactions are common in nuclear physics. When a deuteron (a ^{2}H nucleus) collides with a ^{7}Li nucleus, an α-particle (a ^{4}He nucleus) and a ^{5}He nucleus are formed and a large amount of **binding energy** is converted to kinetic energy. Nuclear physicists refer to this difference in kinetic energy as the **Q-value** of the reaction. Nuclear reactions with zero or negative Q-values also occur. The Q-value is a useful parameter to describe any interaction. It is defined as follows:

$$\text{Q-value} = \Delta E_k = (\sum E_k)_{\text{final}} - (\sum E_k)_{\text{initial}} \qquad 2.6(1)$$

in which the sums of the kinetic energies refer to that of all the bodies involved in the interaction.

Whatever the nature of an interaction (i.e. elastic, endoergic or exoergic), the total momentum will remain unchanged if no unbalanced external forces act on the system. When an interaction occurs in the absence of unbalanced external forces, the conservation of momentum may be used to write down three equations (one for each component) by means of which unknown quantities may be calculated. In some cases it will be found that these three equations are not sufficient to calculate all the unknown quantities. In such cases it will be necessary to know the Q-value of the interaction and the principle of energy conservation will supply one more equation.

If an unbalanced external force is present during an interaction and it is small compared to the forces which the interacting bodies exert on each other and the duration of the interaction is short, the effect of the external force may be disregarded and the conservation of momentum used as a good approximation.

2.6.2 Impulse

Consider the interaction of bodies 1 and 2 of which the masses are m_1 and m_2 respectively. The duration of the interaction is from $t = t_1$ to $t = t_2$ and in that interval of time the velocity of body 1 changes from $\bar{v}_1$ to $\bar{v}_2$ and that of body 2 from $\bar{u}_1$ to $\bar{u}_2$. While the interaction is in progress, very little or nothing is known about the forces which the bodies exert on each other. When two billiard balls collide, the contact time is very short. It is obvious that the magnitude of the forces must be large in order to accomplish the ensuing change in the momentum of each ball in such a short interval of time. Such forces which are strongly time-dependent are called **impulsive forces**. The name may be misleading in that it might create the impression that we are dealing with a new force of which the effects will be different from those which have previously been discussed.

Let $\bar{F}_{21}$ be the force experienced by body 1 and $\bar{F}_{12}$, that by body 2 while the interaction is in progress. In general they will be strongly time-dependent. Although the functions (of t) which describe them will be unknown in most cases, Newton's third law will be valid at each instant:

$$\bar{F}_{21} = -\bar{F}_{12} \qquad 2.6(2)$$

For body 1, Newton's second law is applied as follows:

$$\begin{aligned} \bar{F}_{21} &= m_1 \frac{d\bar{v}}{dt} \\ \text{so that} \quad \bar{F}_{21}\, dt &= m_1\, d\bar{v} \\ \text{and} \quad \int_{t_1}^{t_2} \bar{F}_{21}\, dt &= \int_{v_1}^{v_2} m_1\, d\bar{v} = m_1\bar{v}_2 - m_1\bar{v}_1 \qquad 2.6(3) \end{aligned}$$

The integral on the left, which can seldom be calculated, is known as the *impulse* of the force $\bar{F}_{21}$ on body 1 and is equal to the change in its momentum during the time interval under consideration. The units of impulse are newton-second (N s) and the reader can verify that this is the same as kg m s^{-1}, the units of momentum.

In exactly the same way, it follows for body 2:

$$\int_{t_1}^{t_2} \bar{F}_{12}\, dt = \int_{u_1}^{u_2} m_2\, d\bar{u} = m_2\bar{u}_2 - m_2\bar{u}_1 \qquad 2.6(4)$$

From equations 2.6(2), 2.6(3) and 2.6(4) it follows that:

$$\begin{aligned} m_1\bar{v}_2 - m_1\bar{v}_1 &= -(m_2\bar{u}_2 - m_2\bar{u}_1) \\ \text{so that} \quad m_1\bar{v}_1 + m_2\bar{u}_1 &= m_1\bar{v}_2 + m_2\bar{u}_2 \qquad 2.6(5) \end{aligned}$$

which shows that the principle of conservation of momentum follows directly from Newton's laws of motion. The prerequisite for its use in the solution of a problem is that no unbalanced external forces should act on the system, or, if such forces are present, they should be small compared to the impulsive forces and the interaction time should be short. The nature of the forces, $\bar{F}_{12}$ and $\bar{F}_{21}$, i.e. whether they are conservative or not, and possible conversions of energy during an interaction, have no effect on the principle of conservation of momentum.

2.6.3 Summary of principles which apply when solving collision problems

The principle of momentum conservation may by applied to *all* interactions if (i) no unbalanced external forces are present (ii) unbalanced external forces which are present are much smaller than the impulsive forces and the duration of the interaction is short. The last condition is an approximation. The conservation of momentum means that the resultant of all the momentum vectors before the interaction is equal to the sum of all the momentum vectors after the interaction.

The principle of energy conservation applies to all interactions as follows:

kinetic energy before interaction + Q-value = kinetic energy after interaction

The Q-value ($\Delta E_k = (\sum E_k)_{\text{final}} - (\sum E_k)_{\text{initial}}$) will be negative, zero or positive depending on whether the interaction is endoergic, elastic or exoergic.

Summary:

Elastic interaction:	$\Delta\bar{p} = \bar{0}$	$Q = 0$
Endoergic interactions:	$\Delta\bar{p} = \bar{0}$	$Q < 0$
Endoergic, completely inelastic:	$\Delta\bar{p} = \bar{0}$	$Q < 0,\ \ \bar{u}_2 = \bar{v}_2$
Exoergic interactions:	$\Delta\bar{p} = \bar{0}$	$Q > 0$

2.6.4 The collision parameter

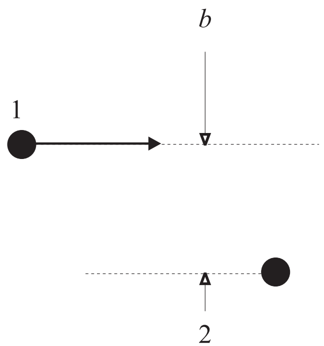


Figure 2.6-1

The **collision parameter**, also known as the **impact parameter** or **collision diameter**, is mainly used in the study of atomic and nuclear physics. Consider a frame of reference in which particle 2 is at rest. Such a frame of reference can always be found by means of a suitable transformation. The length of the perpendicular from the centre of mass of particle 2 to the line through the centre of mass of particle 1 in the direction of its relative velocity vector, is called the collision parameter. It is indicated by b in figure 2.6-1.

If $b = 0$ it will be a **head-on collision**. If b is larger than the sum of the radii of two spherical objects, no collision will take place. If two spheres interact by means of *force at a distance* such as a coulomb-force or gravity, the interaction will take place even if b exceeds the sum of their radii. In such cases the interacting bodies will not touch each other during the interaction. If $b > 0$ and an interaction takes place, it is called an **oblique collision** which is also known as an **off-centre collision**.

2.6.5 Examples involving collisions

1. Body 1 has a mass of 1 kg and a velocity of $4\hat{x}$ $\mathrm{m\,s^{-1}}$. It collides head-on with body 2 which has a mass of 2 kg and an initial velocity of $-3\hat{x}$ $\mathrm{m\,s^{-1}}$. The velocity of body 1 is $\bar{u} = u\hat{x}$ after the collision and that of body 2, $\bar{v} = v\hat{x}$ $\mathrm{m\,s^{-1}}$. Calculate u and v if the collision is (a) elastic, (b) completely inelastic. Calculate the Q-value of the inelastic collision.

Because no unbalanced external forces are present, the total momentum before the collision is equal to that after the collision.

$$\begin{aligned} 1 \times 4\hat{x} + 2(-3\hat{x}) &= 1 \times u\hat{x} + 2 \times v\hat{x} \\ 4\hat{x} - 6\hat{x} &= u\hat{x} + 2v\hat{x} \\ u + 2v &= -2 \qquad \text{............(1)} \end{aligned}$$

This equation is valid for elastic and inelastic collisions.

(a) For an elastic collision, the kinetic energy is conserved.

$$\begin{aligned} \frac{1}{2} \times 1 \times (4)^2 + \frac{1}{2} \times 2 \times (-3)^2 &= \frac{1}{2} \times 1 \times (u)^2 + \frac{1}{2} \times 2 \times (v)^2 \\ u^2 + 2v^2 &= 34 \qquad \text{..........(2)} \end{aligned}$$

The reader can verify the following solutions for u and v from equations (1) and (2).

$$\begin{aligned} v &= -3 \quad \text{or} \quad 5/3 \\ u &= 4 \quad \text{or} \quad -16/3 \end{aligned}$$

The values -3 and 4 are the original values of the velocities and they represent the case in which momentum and energy are conserved without a collision occurring. The correct solutions are thus:

$$v = 5/3 \qquad \text{and} \qquad u = -16/3\ \mathrm{m\,s^{-1}}$$

(b) For a completely inelastic collision, $u = v$. If this is substituted in equation (1), we have:

$$\begin{aligned} 3u &= -2 \\ u = v &= -2/3 \quad \mathrm{m\,s^{-1}} \end{aligned}$$

The Q-value of the inelastic interaction, is the change in kinetic energy.

$$\begin{aligned} Q &= \Delta E_k = (\sum E_k)_{\text{final}} - (\sum E_k)_{\text{initial}} \\ &= \left(\frac{1}{2}\times 1\times(-\frac{2}{3})^2 + \frac{1}{2}\times 2\times(-\frac{2}{3})^2\right) - \left(\frac{1}{2}\times 1\times(4)^2 + \frac{1}{2}\times 2\times(-3)^2\right) \\ &= -16,33 \quad \text{J} \end{aligned}$$

2\.

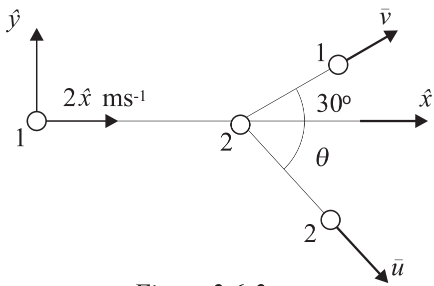


Figure 2.6-2

The sketch in figure 2.6-2 shows two bodies, each with mass 3 kg, involved in an oblique elastic collision on a horizontal surface. The interaction between the surface and the bodies may be disregarded. The initial speed of body 1 is 2 m s^{-1} and body 2 is initially at rest. After the collision, body 1 moves at an angle of 30° to its original direction. Calculate the speeds, v and u, of bodies 1 and 2 respectively and also the angle θ in which 2 moves, after the collision.

Choose a frame of reference as shown in the sketch. This is a two-dimensional problem and the collision can be described fully by using two components only. Since no unbalanced external forces act on the system, the momentum is conserved. From this follows:

$$3\times 2\hat{x} + \bar{0} = 3[(v\cos 30^\circ)\hat{x} + (v\sin 30^\circ)\hat{y}] + 3[(u\cos\theta)\hat{x} - (u\sin\theta)\hat{y}]$$

in which the sum of the x-components on the left-hand side is equated to the sum of the x-components on the right-hand side. Doing the same with the y-components, provides a second equation. From this follows:

$$\begin{aligned} \sqrt{3}v + 2u\cos\theta &= 4 \qquad \text{..........(1)} \\ v - 2u\sin\theta &= 0 \qquad \text{..........(2)} \end{aligned}$$

Since the collision is elastic, the kinetic energy is conserved. From this follows:

$$\begin{aligned} \frac{1}{2}\times 3\times(2)^2 + 0 &= \frac{1}{2}\times 3\times(v)^2 + \frac{1}{2}\times 3\times(u)^2 \\ v^2 + u^2 &= 4 \qquad \text{.............(3)} \end{aligned}$$

The solution of the three simultaneous equations, gives the following answers:

$$v = \sqrt{3} = 1,732 \quad \text{m s}^{-1}, \qquad u = 1 \quad \text{m s}^{-1}, \qquad \theta = 60^\circ$$

(*Comment*: If simultaneous non-linear equations that contain trigonometric ratios, are to be solved, it is advisable to eliminate those ratios first. In this case it can be done as follows: Take the cosine term in equation (1) to the right-hand side and 4 to the left. Take the sine term in equation (2) to the right-hand side. Square both equations and add them. θ is eliminated. The rest of the procedure should be obvious.)

3.

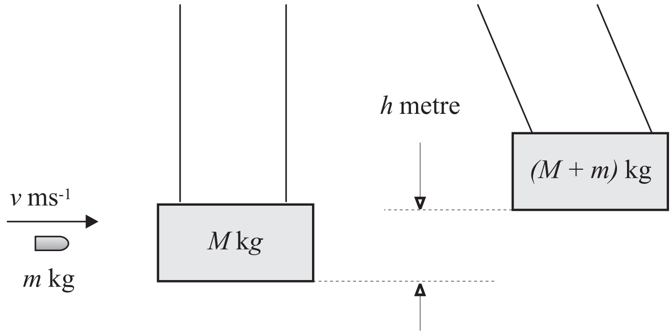


Figure 2.6-3

Figure 2.6-3 shows a **ballistic pendulum** by means of which the speed of a bullet may be determined experimentally. A block of wood with mass M kilogram is suspended by four light cords in such a way that it stays horizontal when it is displaced from the equilibrium position. A bullet with mass m kilogram and horizontal velocity $\bar{v}$ m s^{-1} strikes the block while it is at rest. The bullet enters the block and stops in it. The time taken for the bullet to be stopped in the block may be disregarded, and the common initial velocity of the block with the bullet in it, is $\bar{V}$ m s^{-1}. The block swings upwards and reaches a position h metre vertically above the plane in which it was at rest. Calculate: (a) v in terms of V, (b) $V = V(h)$ and then $v = v(h)$, (c) the Q-value of the interaction in terms of M, m, h and g.

(a) As the bullet penetrates the wood, no unbalanced external forces act on the system. The weights of the bodies are balanced by the tension in the cords. The momentum is thus conserved.

$$m \times \bar{v} + \bar{0} = (M+m)\bar{V}$$

from which

$$v = \frac{M+m}{m} V \quad \text{m s}^{-1}$$

(b) After the bullet comes to rest in the block, they swing upwards against gravity which is conservative. The mechanical energy is therefore conserved. Choose the original position of the block as reference for the calculation of the gravitational potential energy. This gives:

$$\begin{aligned} \frac{1}{2}(M+m)V^2 + 0 &= 0 + (M+m)gh \\ \text{from which follows} \qquad V &= (2gh)^{\frac{1}{2}} \\ \text{and from (a)} \qquad v &= \frac{M+m}{m}(2gh)^{\frac{1}{2}} \end{aligned}$$

(c) The Q-value is given by the change in kinetic energy.

$$\begin{aligned} Q &= \frac{1}{2}(M+m)V^2 - \frac{1}{2}mv^2 \\ &= (M+m)gh - \frac{1}{2}mv^2 \\ &= (M+m)gh - \frac{1}{2}m\frac{(M+m)^2}{m^2}(2gh) \\ &= -\frac{M(M+m)}{m}gh \quad \text{joule} \end{aligned}$$

2.7 Applications: Atomic and nuclear physics

2.7.1 The electron volt

Consider a particle with mass m kilogram and electric charge q coulomb in an electrostatic field. At position 1 in this field the electric potential is V_1 volts and there the particle has a speed of v_1 m s^{-1}. At position 2, the potential is V_2 volts and the speed of the particle, v_2 m s^{-1}. Since the field, which is conservative, is the only interaction with the particle, the energy of the system is conserved.

$$\begin{aligned} (E_k + \text{electric } E_p)_1 &= (E_k + \text{electric } E_p)_2 \\ \frac{1}{2}mv_1^2 + qV_1 &= \frac{1}{2}mv_2^2 + qV_2 \\ \text{so that} \qquad \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 &= q(V_1 - V_2) \end{aligned}$$

The increase (or decrease) in the kinetic energy is equal to the decrease (or increase) in its electrical potential energy.

Consider the special case of a particle which is at rest in position 1 ($v_1 = 0$). In this case we have:

$$\frac{1}{2}mv_2^2 = q(V_1 - V_2)$$

which implies that the particle had accelerated. The kinetic energy cannot be negative (a negative kinetic energy would require an imaginary speed) and therefore $V_1 > V_2$ for a positively charged particle and $V_1 < V_2$ for one that is negatively charged.

If the potential difference, $V_1 - V_2$ is measured in volts (joules/coulomb) and the electric charge in coulomb, the kinetic energy is given in (coulomb)$\times$(volt) = joule. For reasons which will become clear if the following examples are studied, atomic and nuclear physicists and also chemists often prefer to express the charge as a multiple of the magnitude of the electron charge ($e = 1,602177 \times 10^{-19}$ C). The energy unit which follows from this choice, is called the **electron volt (eV)**.

$$\begin{aligned} 1 \text{ eV} = 1 \text{ electron charge } \times 1 \text{ volt} &= 1,602177 \times 10^{-19} \text{ coulomb } \times 1 \text{ volt} \\ &= 1,602177 \times 10^{-19} \text{ joule} \end{aligned}$$

We may define an electron volt (eV) as follows: *If a free electron is accelerated through an electric potential difference of one volt, its kinetic energy changes by one electron volt.*

In X-ray tubes electrons are accelerated through thousands of volts and then their energies are expressed in keV (kiloelectron volt). In large accelerators the particles which are accelerated, may reach energies which are expressed in MeV (megaelectron volt) or even GeV (gigaelectron volt).

The electron volt is not an SI unit but it has so many advantages in certain fields of study that it will probably be kept in use. The following examples illustrate some of the advantages:

If an electron is accelerated from rest through a potential difference of 500 V, it has a kinetic energy of 500 eV.

If a proton (charge equal to that of an electron but opposite in sign) is accelerated through 500 V, it has a kinetic energy of 500 eV.

If an α-particle (two protons and two neutrons) is accelerated through 500 V, it has a kinetic energy of 1000 eV or 1 keV.

If a ^{6}Li-nucleus (three protons and three neutrons) is accelerated through 500 V, it has a kinetic energy of 1,5 keV.

A ^{6}Li-nucleus accelerated through 20 MV, has a kinetic energy of 60 MeV.

Example:

The mass of a proton is $1,67 \times 10^{-27}$ kg. Calculate its speed if its kinetic energy is 3 MeV.

$$3 \quad \text{MeV} \quad = \quad 3 \times 10^6 \quad \text{eV}$$

$$= 3 \times 10^{6} \times 1,602 \times 10^{-19} \quad \text{joule}$$
$$= 4,80 \times 10^{-13} \quad \text{joule}$$

but $$E_k = \frac{1}{2}mv^2$$

so that $$v = [2E_k/m]^{\frac{1}{2}}$$
$$= [2 \times (4,8 \times 10^{-13})/1,67 \times 10^{-27}]^{\frac{1}{2}}$$
$$= 2,40 \times 10^{7} \quad \mathrm{m\,s^{-1}}$$

2.7.2 The dynamics of nuclear reactions

A typical nuclear reaction consists of the collision of particles A and B which combine to form an intermediate nucleus that decays with the formation of particles C and D. In this process nuclear binding energy is converted to kinetic energy in the case of an exoergic reaction (positive Q-value) or the conversion of kinetic energy to nuclear binding energy in the case of an endoergic reaction (negative Q-value). Usually one of the particles A or B moves relative to the laboratory and it is called the **projectile particle** whilst the other, which is at rest, is known as the **target particle**. These names are artificial and have no practical implications.

Both the projectile and the target consist of nucleons (protons and neutrons) and in nuclear reactions nucleons are preserved. This means that the total number of nucleons in A and B are equal to the sum of those in C and D. In general the sum of the masses of A and B will differ from the sum of the masses of C and D by a relatively small amount. The mass difference is equivalent to the Q-value according to the Einstein formula in equation 2.5(21):

$$(\Delta m)c^2 = Q \qquad 2.7(1)$$

in which Δm is the mass difference and c the speed of light in free space. Due to the fact that c^2 is very large, even a small mass difference will be equivalent to a relatively large amount of energy.

Since every equation for momentum conservation is **homogeneous** in mass (*each term contains mass to the first power*), the units in which the mass is measured do not influence the validity of the momentum equation. Since the relative error is small, nuclear physicists prefer to use the **mass number** as unit of mass when applying the principle of momentum conservation. The difference between the masses of a proton and a neutron is disregarded and the mass number is equal to the number of nucleons in the particle. The mass number has the advantage that it is always an **integer**.

The velocities of nuclear particles are generally not of interest because their energies can be measured directly by means of **particle detectors**. For this reason a very convenient relationship which expresses the magnitude of the momentum of a particle in terms of its kinetic energy, is often used.

$$\text{Since} \qquad E_k = \frac{1}{2}mv^2 \qquad \text{and} \qquad p^2 = m^2v^2 = 2mE_k$$

$$\text{it follows that} \qquad p = (2mE_k)^{\frac{1}{2}} \qquad 2.7(2)$$

Since the equations for the conservation of energy are homogeneous in energy, their validity is not influenced by the units in which the energy is measured. In nuclear physics, multiples of electron volt are used, usually MeV.

Example:

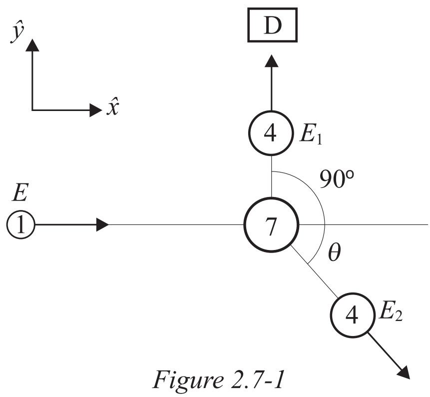


Figure 2.7-1

A proton is accelerated by a potential difference of 2 MeV in a Van de Graaff accelerator. In the reaction chamber it hits a ^{7}Li target and the intermediate nucleus ^{8}Be is formed. The latter decays immediately to form two α-particles (nuclei of ^{4}He). The Q-value of the reaction is 17,348 MeV. One of the α-particles is recorded by a detector situated at right angles to the incident proton beam. Calculate the energies of both α-particles and the direction in which the one which is not detected, moves. The experimental set-up is shown in figure 2.7-1. D represents the detector.

The charge of a proton has the same magnitude as that of an electron. If the proton is accelerated through 2 MV, it has a kinetic energy of 2 MeV. Choose a frame of reference with origin at the target, $\hat{x}$ in the direction of the proton beam and $\hat{y}$ in the direction of the detector as shown in the sketch. Using equation 2.7(2), the momentum of an incident proton may be written as follows:

$$\bar{p} = (2 \times 1 \times E)^{\frac{1}{2}}\hat{x} = (2E)^{\frac{1}{2}}\hat{x}$$

If the energy of the detected α-article is E_1, its momentum is given by:

$$\bar{p}_1 = (2 \times 4 \times E_1)^{\frac{1}{2}}\hat{y} = (8E_1)^{\frac{1}{2}}\hat{y}$$

If the energy of the other α-particle is E_2 and its egression angle, θ as shown in the sketch, its momentum is given by:

$$\bar{p}_2 = [(8E_2)^{\frac{1}{2}}\cos\theta]\hat{x} - [(8E_2)^{\frac{1}{2}}\sin\theta]\hat{y}$$

Since no unbalanced external forces are present, momentum is conserved. (The weights of the particles have an unmeasurable effect during the short duration of the interaction. In the example in 2.7.1 it was calculated that the speed of the proton is of the order 10^7 m s^{-1}.)

$$\begin{aligned}\bar{p} &= \bar{p}_1 + \bar{p}_2 \\ (2E)^{\frac{1}{2}}\hat{x} &= (8E_1)^{\frac{1}{2}}\hat{y} + [(8E_2)^{\frac{1}{2}}\cos\theta]\hat{x} - [(8E_2)^{\frac{1}{2}}\sin\theta]\hat{y}\end{aligned}$$

By first equating the x-components and then the y-components, the following two equations are obtained after simplification:

$$2\sqrt{E_2}\cos\theta = \sqrt{E} \quad \text{...........(1)}$$
$$\sqrt{E_2}\sin\theta = \sqrt{E_1} \quad \text{..........(2)}$$

From the principle of energy conservation, we have the following:

$$E_1 + E_2 = E + Q \quad \text{..........(3)}$$

in which $E = 2$ MeV and $Q = 17{,}348$ MeV.

The following answers were calculated from the above equations:

$$E_1 = 9,424 \text{ MeV} \qquad E_2 = 9,924 \text{ MeV} \qquad \theta = 77{,}03°$$

2.8 Relativistic dynamics

2.8.1 Rest mass and relativistic mass

In section 1.8 the special theory of relativity was discussed. The following constant occurs frequently in this theory:

$$\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$$

The reader can verify the fact that this constant tends to infinity when the relative speed, v tends to c, the speed of light in free space. A relative speed in excess of c, makes γ imaginary and the Lorentz transformation would then give imaginary components for position and velocity which are physically unacceptable. A relative speed in excess of light speed in free space, is just not possible.

The so-called summation rule for velocities is given in simplified form by equation 1.8(17). In effect this rule says that the speed of light "plus" any other speed, is equal to the speed of light. The plus sign is written between inverted commas because it does not represent a summation as is the case in classical relativity but rather a calculation which agrees with the Lorentz transformation. This result confirms the fact that no body can exceed the speed of light in free space relative to any observer.

This important fact has serious implications regarding Newton's second law. If a force $\bar{F}$ acts on a body of mass m, its momentum which is given by $\bar{p} = m\bar{v}$, will change in accordance with Newton's second law:

$$\bar{F} = \frac{d\bar{p}}{dt}$$

If a constant force, $\bar{F}$ acts for a time interval Δt, the impulse which is equal to the change in momentum, is given by:

$$\int_0^{\Delta t} \bar{F}\,dt = \bar{F}\Delta t = \Delta\bar{p} = m(\Delta\bar{v})$$

From this result it would seem that if the time interval is long enough, the change in momentum could be large enough for the speed of the body to exceed that of light in free space. According to classical physics, this should be possible but that would violate the relativity theory which gives an upper limit of c to the speed of any object. This contradiction shows that either Newton's second law or the theory of relativity is incorrect. Up to this stage it has been assumed that the mass of a body is Lorentz-invariant. This is in fact incorrect and is the reason for the apparent contradiction. The contradiction disappears if the mass in the expression for momentum is replaced by the following:

$$m = \gamma m_0 = m_0(1 - v^2/c^2)^{-\frac{1}{2}} \qquad 2.8(1)$$

which is known as the **relativistic mass** of the body. The factor m_0, is called its *rest mass*, which is the mass that would be measured if the body is at rest relative to the observer.

As seen in table 1.8-1, the factor γ increases with increasing relative speed, v. The relativistic mass of a body will consequently also increase with increasing speed relative to the observer. Figure 2.8-1 shows a graphic representation of the relativistic mass as a function of the relative speed. The mass is expressed as multiples of the rest mass and the relative speed as a fraction of the speed of light in free space. From the graph it can be seen that the rest mass will be a good approximation for the relativistic mass if the relative speed is much less than the speed of light in free space.

The gradient of the graph is given by the derivative of equation 2.8(1):

$$\frac{dm}{dv} = \frac{d}{dv}(\gamma m_0) = m_0 \frac{d}{dv}(1 - v^2/c^2)^{-\frac{1}{2}}$$

$$= \frac{m_0 v/c^2}{(1 - v^2/c^2)^{\frac{3}{2}}} = \frac{mv/c^2}{1 - v^2/c^2} \qquad 2.8(2)$$

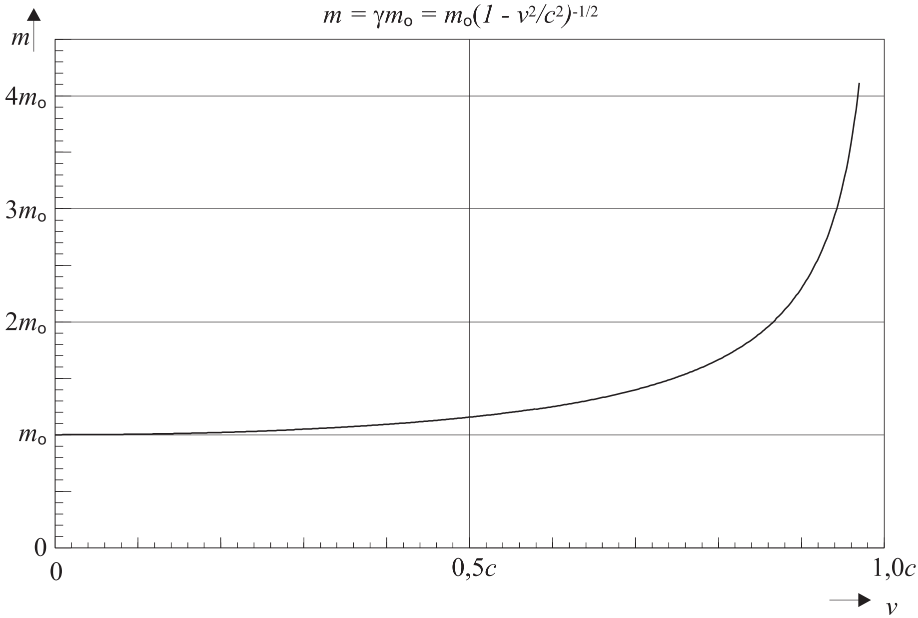


Figure 2.8-1

The dependence of the relativistic mass on the speed, is also demonstrated well by the binomial expansion of the factor γ as follows:

$$\gamma = (1 - v^2/c^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \ldots.. \qquad 2.8(3)$$

from which it can be seen that the higher-order terms become insignificant if v is much smaller than c. At very small values of v, only the first term need be taken into account and that explains why the relativistic mass may then be approximated very well by the rest mass.

Examples:
1. Calculate the relativistic mass of a body with rest mass 1 kg when it moves at a speed of (a) 0,1000c, (b) 0,6000c, (c) 0,9999c relative to an observer.

In table 1.8-1 the values of γ for these speeds are given.

(a). For $v = 0,1000c$, $\gamma = 1,00504$ so that $m(0,1000c) = 1,00504$ kg
(b). For $v = 0,6000c$, $\gamma = 1,25000$ so that $m(0,6000c) = 1,25000$ kg
(c). For $v = 0,9999c$, $\gamma = 70,71245$ so that $m(0,9999c) = 70,71245$ kg

2. The relativistic mass of a body is double its rest mass. Calculate its speed relative to the observer.

$$
\begin{aligned}
2m_0 &= m_0(1 - v^2/c^2)^{-\frac{1}{2}} \\
4 &= (1 - v^2/c^2)^{-1} \\
0,25 &= 1 - v^2/c^2 \\
v^2/c^2 &= 0,75 \\
v &= \sqrt{0,75}c = 0,8660c
\end{aligned}
$$

2.8.2 Relativistic momentum and energy

The use of relativistic mass instead of rest mass, eliminates the apparent discrepancy between the classical and relativistic dynamics. This can be proved by applying the Lorentz transformation to the principle of momentum conservation. The correct definition for momentum is thus:

$$\bar{p} = m\bar{v} = \gamma m_0 \bar{v} = m_0 \bar{v}(1 - v^2/c^2)^{-\frac{1}{2}} \qquad 2.8(4)$$

This quantity is known as the **relativistic momentum**. Figure 2.8-2 shows how the magnitudes of the classical momentum and the relativistic momentum

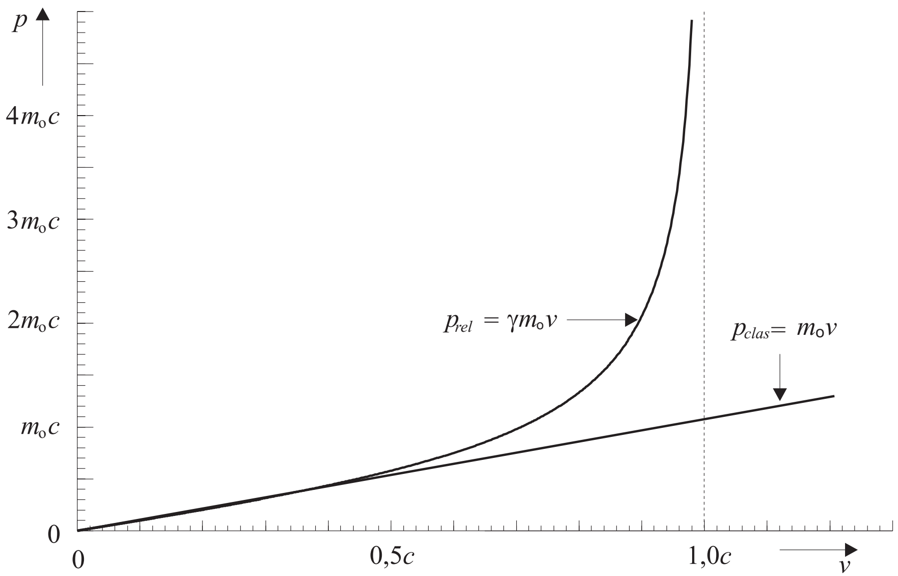


Figure 2.8-2

vary with speed relative to the observer. The momentum scale is in units of $m_0 c$ and the speed scale in fractions of c.

The graph shows that the relativistic momentum may be approximated by the classical momentum when the speed is much less than the speed of light in free space.

When a force $\bar{F}$ acts on a mass m, its momentum changes according to Newton's second law:

$$\bar{F} = \frac{d}{dt}(m\bar{v})$$

Since mass is a function of speed, the expression does not simply reduce to the well-known equation $\bar{F} = m\bar{a}$ which is used in classical dynamics. The time-derivative of the relativistic momentum is given by:

$$\bar{F} = m\frac{d\bar{v}}{dt} + \bar{v}\frac{dm}{dt} = m\bar{a} + \bar{v}\frac{dm}{dt} \qquad 2.8(5)$$

from which follows some rather surprising results: The acceleration is generally not parallel to the force which causes it and the magnitude of the force is not proportional to that of the acceleration. Such a proportionality does exist in classical dynamics and the proportionality constant is the mass.

For the special case where the force is perpendicular to the velocity vector, it cannot change the speed and therefore, neither the relativistic mass. In such a case, the second term in 2.8(5) is zero. Then:

$$F = ma = m\frac{v^2}{r} = m_0\frac{v^2}{r}(1 - v^2/c^2)^{-\frac{1}{2}} \qquad (\bar{F}\perp\bar{v}) \qquad 2.8(6)$$

This result is of prime importance in particle accelerators in which the particles move on circular orbits or portions of circular orbits such as the **synchrocyclotron** and the **synchrotron**.

If the force is parallel to the velocity vector as is the case in rectilinear acceleration, equation 2.8(5) becomes:

$$F = ma + v\frac{dm}{dt} = ma + v\frac{dm}{dv}\frac{dv}{dt} = (m + v\frac{dm}{dv})a$$

If use is made of the expression for dm/dv in 2.8(2), this relationship reduces to:

$$F = m_0(1 - v^2/c^2)^{-\frac{3}{2}}a \qquad (\bar{F}\|\bar{v}) \qquad 2.8(7)$$

Here the magnitude of the force exceeds ma and it tends to infinity when $v \to c$. In effect this means that if a body moves at the speed of light relative to an observer, it would require an infinitely large force for a further increase of speed.

This is in agreement with the fact that the speed of any mass cannot exceed that of light in free space.

For the calculation of the kinetic energy, E_k, consider a body which is initially at rest and then make a calculation similar to that in 2.5.1 (the classical case) with the necessary relativistic adaptations.

$$E_k = \int_0^s \bar{F} \cdot d\bar{s} = \int_0^t \bar{F} \cdot \frac{d\bar{s}}{dt} dt = \int_0^t \bar{F} \cdot \bar{v}\, dt$$

in which $\bar{F} \cdot \bar{v}$ is the power of the force, and the integral the work done during the specified time interval.

If the expression in 2.8(5) is used for the force vector, the following is obtained:

$$\bar{F} \cdot \bar{v} = m\frac{d\bar{v}}{dt} \cdot \bar{v} + \bar{v} \cdot \bar{v}\frac{dm}{dt} = m\frac{d\bar{v}}{dt} \cdot \bar{v} + v^2\frac{dm}{dt}$$

As in 2.5.1 The first term may be rewritten as follows:

$$m\frac{d\bar{v}}{dt} \cdot \bar{v} = m\frac{d}{dt}(\frac{1}{2}v^2) = mv\frac{dv}{dt}$$

with which the expression for the power, $\bar{F} \cdot \bar{v}$, may be reduced in the following manner:

$$\bar{F} \cdot \bar{v} = mv\frac{dv}{dt} + v^2\frac{dm}{dt} = mv\frac{dv}{dm}\frac{dm}{dt} + v^2\frac{dm}{dt} = (mv\frac{dv}{dm} + v^2)\frac{dm}{dt}$$

The derivative dv/dm is the reciprocal of dm/dv which was calculated in 2.8(2). If the value is substituted in the above equation, it reduces to the following simple expression:

$$\bar{F} \cdot \bar{v} = c^2\frac{dm}{dt}$$

Since the body is accelerated from rest, the initial value of the relativistic mass is equal to the rest mass, m_0. If the mass at time t is equal to $m = \gamma m_0$, the kinetic energy is given by the following:

$$E_k = \int_0^t \bar{F} \cdot \bar{v} dt = \int_0^t c^2\frac{dm}{dt} dt = \int_{m_0}^m c^2 dm = (m - m_0)c^2$$

This is one of the very important results of the special theory of relativity: *The kinetic energy of a body is equal to the difference between its relativistic mass and its rest mass multiplied by the square of the speed of light in free space.*

$$E_k = (m - m_0)c^2 = m_0(\gamma - 1)c^2 \qquad 2.8(8)$$

By using the binomial expansion of γ, the kinetic energy may be written as an infinite series as follows:

$$\begin{aligned} E_k &= m_0 c^2 (1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots\ldots - 1) \\ &= \frac{1}{2} m_0 v^2 (1 + \frac{3}{4}\frac{v^2}{c^2} + \ldots\ldots) \end{aligned}$$

from which it can be seen that the classical expression for kinetic energy, $E_k = \frac{1}{2}mv^2$ is an excellent approximation for the relativistic value when the speed of the body is much less than that of light in free space.

Further considerations which are corroborated by experimental evidence concerning nuclear reactions, lead to the conclusion that *all* mass is associated with all forms of energy. The mass of a body increases if its energy is increased, be it chemical, potential, kinetic, electrical, thermal or whatever form. If the energy of a body increases, its mass increases and if it diminishes, its mass diminishes. The relationship between the change in energy, ΔE, and the corresponding change in mass, Δm, is given by:

$$\Delta E = \Delta m\, c^2 \qquad 2.8(9)$$

Energy and mass are thus only different ways in which the same physical quantity is measured and the relationship between kilogram and mass is given by the following:

$$1 \text{ kg} = c^2 \text{ joule} = 8,99 \times 10^{16} \text{ joule}; \qquad 1 \text{ joule} = c^{-2} \text{ kg} = 1,11 \times 10^{-17} \text{ kg}$$

The magnitudes of these numbers should make it clear why it is impossible to detect differences in mass due to energy changes in everyday situations. Lifting a mass of 1 kg to a height of 1000 m at constant velocity, would increase its gravitational potential energy by about 10^4 J. This would cause the combined mass of the earth and the mass of 1 kg to increase by about 10^{-13} kg. In nuclear reactions the energies which are involved are relatively large and the corresponding changes in mass are measured quite easily. It is thus known that an amount of energy, E, is associated with a mass m and vice versa according to the relationship $E = mc^2$.

By the identification of mass and energy, the laws of conservation of mass and conservation of energy, become one combined law. The total energy of a body with mass relativistic m, is thus equal to mc^2. From equation 2.8(8) follows:

$$\text{total energy} = E = mc^2 = m_0 c^2 + E_k \qquad 2.8(10)$$

The term $m_0 c^2$, is called the **rest energy** which a body possesses because it possesses rest mass.

Another useful expression for the total energy of a body, expresses it in terms of its relativistic momentum and is analogous to equation 2.7(2) in classical dynamics.

$$E = mc^2 = m_0 c^2 (1 - v^2/c^2)^{-\frac{1}{2}} = (m_0^2 c^4 + p^2 c^2)^{\frac{1}{2}} \qquad 2.8(11)$$

The **photon** (a quantum of light) and the **neutrino** are examples of particles which do not possess rest mass, and can exist only whilst moving at the speed of light. They do, however, possess momentum. The change in momentum of photons is observed in the phenomenon of **radiation pressure** which is partially responsible for the formation of the tails of comets which always point away from the sun.

Particles which possess rest mass can never attain the speed of light in free space since it would require an infinite amount of energy. It is , however, possible for a particle to exceed the speed of light in a *material medium*. When an electrically charged particle exceeds the speed of light in that medium, a conical wave front of light (called **Cerenkov radiation**) is emitted. It is formed in the same manner as the sonic shock wave emitted by an aircraft which exceeds the speed of sound in air or the formation of the bow wave of a water-craft which exceeds the speed of surface waves on water. (See chapter 5: The Doppler effect).

2.8.3 The atomic mass and its energy equivalent

The **atomic mass unit** (abbreviation: u) is often used in particle physics to specify the masses of atoms and atomic nuclei. It is defined as one twelfth of the mass of a neutral atom of the most abundant isotope of carbon, namely ^{12}C of which the mass is chosen as 12 u exactly. From this follows:

$$\begin{aligned} 1 \text{ u} &= \frac{1}{12}(\text{mass of one } ^{12}\text{C atom}) \\ &= \frac{1}{12}\left(\frac{1,2 \times 10^{-2}}{6,02252 \times 10^{23}}\right) = 1,66043 \times 10^{-27} \text{ kg} \end{aligned}$$

The joule-equivalent of 1 u may be calculated by multiplying its kilogram-equivalent by the square of the speed of light in free space.

$$\begin{aligned} 1 \text{ u} &= (1,66043 \times 10^{-27}) \times (2,99792 \times 10^8)^2 \\ &= 1,49232 \times 10^{-10} \quad \text{joule} \\ \text{but} \quad 1 \text{ eV} &= 1,602177 \times 10^{-19} \quad \text{joule} \\ \text{so that} \quad 1 \text{ u} &= \frac{1,49232 \times 10^{-10}}{1,602177 \times 10^{-19}} = 9,314 \times 10^8 \quad \text{eV} = 931,4 \quad \text{MeV} \end{aligned}$$

For the calculation of Q-values (energy liberation or energy absorption) in nuclear reactions, nuclear physicists often express the masses of particles in terms of their energy equivalents. For this purpose MeV is often used since many accelerators operate in this energy range.

Examples:

1. Calculate the rest energy of an electron. Then calculate the energy of a γ-quantum which will be able to create an electron-positron pair in a lead plate in such a way that they are at rest. The rest mass of a positron is the same as that of an electron and equal to $9,108 \times 10^{-31}$ kg. The speed of light in free space is 3×10^8 m s^{-1}. Give the answer in MeV.

The rest energy of an electron or a positron is given by:

$$\begin{aligned} E_0 &= m_0 c^2 = (9,108 \times 10^{-31}) \times (3 \times 10^8)^2 \\ &= 8,197 \times 10^{-14} \quad \text{joule} \\ &= (8,197 \times 10^{-14}) \div (1,602 \times 10^{-19}) \quad \text{eV} = 5,177 \times 10^5 \quad \text{eV} \\ &= 0,5117 \quad \text{MeV} \end{aligned}$$

The energy of the γ-quantum must be equal to the rest mass of the pair.

$$E_\gamma = 2E_0 = 2(0,5117) = 1,023 \quad \text{MeV}$$

2. The rest mass of an electron is $m_0 = 9,108 \times 10^{-31}$ kg and its charge is $e = -1,602 \times 10^{-19}$ C. An electron is accelerated from rest through an electric potential difference of 10^8 volts. Calculate (a) the total energy of the electron, (b) its relativistic mass, and (c) the speed with which it leaves the accelerator.

(a) The kinetic energy of the electron when it leaves the accelerator is:

$$E_k = 10^8 \quad \text{eV} = 10^2 \quad \text{MeV} = 1,602 \times 10^{-11} \quad \text{J}$$

The total energy of the electron after the acceleration is:

$$\begin{aligned} E &= E_k + m_0 c^2 \\ &= 1,602 \times 10^{-11} + (9,108 \times 10^{-31}) \times (3 \times 10^8)^2 \\ &= 1,610 \times 10^{-11} \quad \text{J} \end{aligned}$$

(b) But

$$\begin{aligned} E &= mc^2 \\ \text{so that} \quad m &= E/c^2 \\ &= 1,789 \times 10^{-28} \quad \text{kg} \\ &= 196,4 \quad \text{electron rest masses} \end{aligned}$$

(c)

$$\begin{aligned} m &= m_0(1 - v^2/c^2)^{-\frac{1}{2}} \\ 196,4 &= (1 - v^2/c^2)^{-\frac{1}{2}} \\ 1 - v^2/c^2 &= 2,592 \times 10^{-5} \\ \text{from which follows} \qquad v &= 0,999987c \end{aligned}$$

3. Calculate the Q-value of the nuclear reaction ^{11}B(p, α)^{8}Be. The masses of the particles involved, are as follows: ^{11}B: 11,00931 u; ^{1}H (a proton): 1,00782 u; ^{4}He (an α-particle): 4,00260 u; ^{8}Be: 8,00531 u.

The Q-value is the sum of the masses of the incident particle and the target minus the sum of the masses of the products.

$$\begin{aligned} Q &= (11,00931 + 1,00782) - (4,00260 - 8,00531) \\ &= 0,00922u \\ &= 0,00922 \times 931,49 = 8,588 \quad \text{MeV} \end{aligned}$$

The fact that the answer is positive, means that the reaction is exoergic, i.e. the products have more kinetic energy than the initial particles when they enter the interaction. When the Q-value for a given reaction is negative, it is endoergic. In an endoergic reaction the end products have less kinetic energy than those which enter the reaction. In such a case the initial particles have to possess a certain minimum energy (called the **threshold energy**) to enable the reaction to take place.

2.9 PROBLEMS: CHAPTER 2

1. A constant force changes the velocity of a mass of 5 kg from $4\hat{r}$ m s^{-1} to $-6\hat{r}$ m s^{-1} in 4 seconds. Calculate the change in momentum and also the force.

2. A motor-car with mass 1500 kg moves at 30 m s^{-1} along a straight line on a horizontal plane. Calculate the magnitude of the force required to bring it to rest it over a distance of 150 m.

3. A force of $0{,}15\hat{z}$ N acts on a body of 0,5 kg while it is displaced by $2\hat{z}$ m. The initial velocity is $0{,}2\hat{z}$ m s^{-1}. Calculate the final velocity and the change in velocity of the body.

4. A vertical force of $25\hat{z}$ N is applied to a body of mass 2 kg which is initially at rest in the gravitational field of the earth. $\bar{g} = -10\hat{z}$ m s^{-2}. The force acts until the displacement is $500\hat{z}$ m. Calculate (a) the resultant force on the mass, (b) the acceleration, (c) the final velocity.

5. A force of $12\hat{x}$ N acts on a body with mass 4 kg. The initial position of the body is $\bar{r}(0) = 7,5\hat{x}$ m and its initial velocity, $\bar{v}(0) = -9\hat{x}$ m s^{-1}. Calculate (a) the acceleration of the body, (b) its velocity as a function of time, (c) its position as a function of time, (d) the instant when it passes through the origin, (e) its momentum when it is at the origin.

6. A force $\bar{F} = 12\hat{x} + 6\hat{y} - 4\hat{z}$ N acts on a body of mass 2 kg of which the initial position and velocity are given by $\bar{r}(0) = 3\hat{x} + 2\hat{y} + \hat{z}$ m and $\bar{v}(0) = -2\hat{x} - 2\hat{y} - \hat{z}$ m s^{-1} respectively. Calculate (a) the acceleration, (b) the magnitude of the acceleration, (c) the velocity as a function of time, (d) the momentum as a function of time, (e) the speed as a function of time, (f) the initial speed, (g) the position vector as a function of time. (h) what distance the body is from the origin at $t = 0$? (i) what the magnitude is of the force which acts on the body?

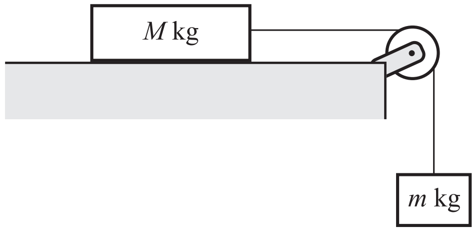
7. A mass of 2 kg is suspended from a light cord. $g = 9,8$ m s^{-2}. Calculate the weight of the mass. Also calculate the tension in the cord when the body accelerates at 5 m s^{-2} upwards and also when it accelerates at 5 m s^{-2} downwards.

8. An elevator with mass 10^3 kg descends in the shaft at an acceleration of 2 m s^{-2}. Calculate the tension in the cable from which it is suspended. A man with mass 100 kg stands on a scale in the elevator. Calculate the reading on the scale while the elevator descends as described above.

9. Two masses of 1,1 kg and 0,9 kg are connected by a light cord over a pulley of which the mass and the friction may be disregarded. Calculate the acceleration

of the system if $g = 10\ \mathrm{m\,s^{-2}}$. If the mass of 1,1 kg descends from rest for 2 seconds before it strikes the ground, calculate how far the 0,9 kg will move upwards before it comes to rest.

10. A mass of M kilogram is connected to a mass of m kg by a light cord over a pulley as shown in the sketch below. Friction may be disregarded. The cord is not stretched in the experiment. Why is the magnitude of the acceleration of the masses the same? Calculate the magnitude of the acceleration and the tension in the cord. Calculate the acceleration in the case where $m \gg M$.



11.The M kg in the previous problem is connected on opposite sides to masses m_1 and m_2 respectively by means of light cords over light pulleys ($m_1 > m_2$). Friction may be disregarded. Calculate the magnitude of the acceleration and the tensions in the cords.

12. A jet of liquid with cross-sectional area A square metres strikes a steel plate at right angles at a speed of $v\ \mathrm{m\,s^{-1}}$. After the liquid has struck the plate, it moves randomly in all directions parallel to it. The density of the liquid is $\rho\ \mathrm{kg\,m^{-3}}$. Calculate the force that the jet exerts on the plate and also the pressure.

13. The constant in Newton's law of gravitation is $G = 6,67 \times 10^{-11}\ \mathrm{N\,m^2kg^{-2}}$. Calculate the force of attraction between the earth, mass $6,0 \times 10^{24}$ kg and a mass of (a) 1 kg, (b) 10^3 kg, (c) 10^6 kg which are at a distance of $6,4 \times 10^6$ m (approximately equal to the radius of the earth) from the centre of mass of the earth. Calculate in each case the accelerations of the objects and that of the earth.

14. The speed of light in free space is $3 \times 10^8\ \mathrm{m\,s^{-1}}$. The mass of the sun is 333 000 times that of the earth. α-Centauri is the nearest star to the sun and its mass is about four times that of the sun. Calculate the gravitational force between the sun and α-Centauri and compare it to the gravitational force between the sun and the earth. The average distance between the sun and the

earth is $1,5 \times 10^{11}$ m. α-Centauri is 4,4 light years from the sun.

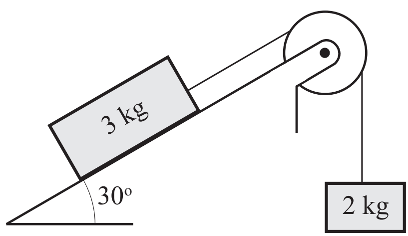
15. The electric charge of an electron is $-1,6 \times 10^{-19}$C and its mass, $9,1 \times 10^{-31}$ kg. An electron is initially at rest when an electric field, $\bar{E} = 7,5 \times 10^3 \hat{x}$ N C^{-1}, is switched on. Calculate the final velocity if the electron moves 5 mm as a result of the action of this field.

16. Two positive electric point charges, each with magnitude $+q$ coulomb are situated on the x-axis of a Cartesian frame of reference at positions $x = +a$ and $x = -a$ m respectively. (a) Calculate the electrostatic field strength, $\bar{E}$, caused by this pair of charges at position $\bar{r} = y\hat{y}$. (b) Calculate the extreme values of the function $E = E(y)$ and sketch its graph in the region $-4a \leq y \leq +4a$. (c) Calculate the force that a charge Q C will encounter at position $\bar{r} = y\hat{y}$ m.

17. Two small polystyrene balls each have a mass of 0,1 g and they carry equal electric charges. They are suspended from two silk threads of length 100 mm each and of which the top ends are fixed at a common position. As a result of electric repulsion, the two threads make an angle of 10° with each other. Calculate the electric charge on each ball.

18. An iron ingot with mass 100 kg rests on a rough horizontal plane. The static coefficient of friction between iron and the plane is 0,4. Calculate the smallest force necessary to set the ingot in motion if the direction of the force is (a) horizontal, (b) upwards at an angle of 30° to the horizontal, (c) downwards at an angle of 30° to the horizontal.

19.



The 2 kg mass in the sketch is in critical equilibrium with downward motion imminent. $g = 10$ m s^{-2}. Calculate the coefficient of static friction between the 3 kg mass and the inclined plane on which it rests. Assume that the coefficients of static and kinetic friction are equal for this problem. Calculate the magnitude of the acceleration if the 2 kg is replaced by a 4 kg mass. Calculate the tension in the cord also while the system accelerates. Friction in the pulley may be disregarded.

20. A body with a mass of 2 kg is projected upwards along a rough inclined plane of which the inclination angle is 30°. The initial speed is 22 m s^{-1}, and the coefficient of kinetic friction is 0,3. (a) Calculate the frictional force while the body moves upwards. (b) What distance will the body move before coming to rest? (c) How long does it move before coming to rest? (d) How long will

it take to slide back from its highest position to the position from where it was projected upwards along the plane? (e) At what speed will it arrive at the initial position? (f) If the mass of the body were 5 kg instead of 2 kg, how would this influence the previous answers?

21. The point of application of the force $\bar{F} = 12,5\hat{x}$ N, has a displacement of $6\hat{x}$ m in 3 seconds. Calculate the work done by the force and also the average power.

22. A force of 12,5 N acts in direction N 30°E. Its point of application is displaced 6 m east in 3 s. Calculate the work done by the force and also the average power.

23. A body is displaced along a straight line from the origin to the position $\bar{r} = 3\hat{x} - 5\hat{y} + 2\hat{z}$ m by the force $\bar{F} = 2\hat{x} + \hat{y} + \hat{z}$ N. Calculate the work done by the force.

24. A particle is displaced along a straight line from position $\bar{r}_1 = 2\hat{x} + 7\hat{y} - 3\hat{z}$ m to position $\bar{r}_2 = 5\hat{x} - 3\hat{y} - 6\hat{z}$ m. During the displacement a force $\bar{F} = 2\hat{x} - 3\hat{y} + 1,5\hat{z}$ N acts on the body. Calculate the work done by the force.

25. A force, $\bar{F} = 20\hat{x} - 10\hat{y} - 15\hat{z}$ N acts on a particle of which the velocity is $\bar{v} = 5\hat{x} - 3\hat{y} + 6\hat{z}$ m s^{-1}. Calculate the power of the force.

26. A man pulls a 50 kg sledge 25 m across a horizontal plane by means of a rope which makes an angle of 30° with the plane. The coefficient of kinetic friction is 0,20. Calculate the work that the man does. $g = 10$ m s^{-2}.

27. The point of application of a force is restricted to a straight line which lies east-west. The magnitude of the force depends on the position of the point of application and is given by $F = (2x - 1)$ newton in which the position from the origin, x, is measured in metre. The direction of the force is constantly E 60°N. Calculate the work done by the force during the following displacements of its point of application: (a) From the origin ($x = 0$) to x = 4 m east. (b) From $x = 2$ m to $x = 6$ m.

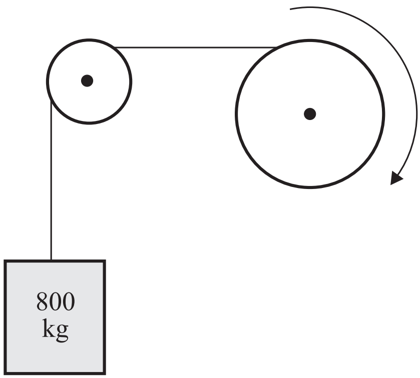
28. When a given helical spring is extended, the force required is proportional to the extension, z, as follows: $\bar{F} = 50z\hat{z}$ in which z is measured in metre. Calculate the work done when the spring is extended from (a) $\bar{r} = \bar{0}$ to $\bar{r} = 3\hat{z}$ m, (b) $\bar{r} = 3\hat{z}$ m to $\bar{r} = 6\hat{z}$ m, (c) From $\bar{r} = 6\hat{z}$ m to $\bar{r} = \hat{z}$ m. What meaning do you ascribe to the sign of the last answer?

29. If the spring in problem 28 were stretched at a constant velocity of $2\hat{z}$, calculate the power at $t = 3$ s. Take $z(0) = 0$.

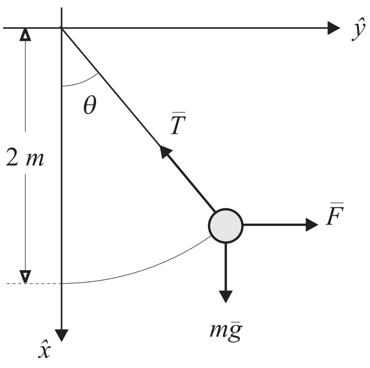
30. Use Newton's law of gravitation to express the weight of a body as a function of the distance between the centres of mass of the earth and the body.

(See equation 2.2(3)). Calculate the work done to remove a body from the earth's gravitational field.

31. An elevator with mass 800 kg is hoisted at a constant speed of 2 $\mathrm{m\,s^{-1}}$ up a mine shaft which is 300 m deep. The cable from which the elevator is suspended has a mass per unit length of 2,5 $\mathrm{kg\,m^{-1}}$. x represents the length of cable taken in as the elevator ascends. If the elevator is at its deepest, $x = 0$ and when it is at the top of the shaft, $x = 300$ m. Write the force which is required to hoist the elevator in this manner as a function of x. Use the line integral of the force vector to calculate the work necessary to hoist the elevator from the bottom to the top of the shaft. Calculate the power when the elevator is 280 m from the top.



32. A mass of 2,5 kg is suspended from a light cord of which the length is 2 m. From the equilibrium position it is displaced very slowly at a constant speed by a horizontal force. Use the line integral of the force vector to calculate the work done to raise the mass 0,2 m above the horizontal plane in which it was in equilibrium. Also calculate the increase in its potential energy and compare it with the work done.



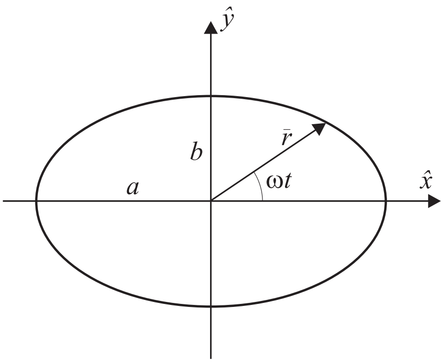
Hint: Method 1: Write $F = F(\theta)$, $|d\bar{r}| = ds = l d\theta$. $\bar{F} \cdot d\bar{r} = \bar{F} \cdot d\bar{s} = F\, ds \cos\theta = F(\cos\theta) l d\theta$.
Method 2: Write $\bar{F}$ and $d\bar{r}$ in terms of $\hat{x}$ and $\hat{y}$. On the circle, $x = l\cos\theta$ and $y = l\sin\theta$. Assume that the mass moves so slowly that $\Delta E_k = 0$.

33. The centre of mass of a body with mass 2 kg, moves along a space curve in a force field. The position vector is given as a function of time by the following:

$$\bar{r} = (2t^3 + t)\hat{x} + (3t^4 - t^2 + 8)\hat{y} + (-12t^2)\hat{z} \quad \text{metre}$$

in which the time, t, is measured in seconds. Calculate (a) the work done by the force field on the body when it moves from $\bar{r}_1 = 8\hat{y}$ m to $\bar{r}_2 = 18\hat{x} + 52\hat{y} - 48\hat{z}$ m, (Before calculating the work, test whether these positions lie on the specified space curve.) (b) the average power exerted by the force during the motion between the two positions, (c) the power at instant $t = 1$ s.

34. A particle with mass m kilogram moves in the plane $z = 0$ of a Cartesian frame of reference and its position vector is given by:



$$\bar{r} = (a \cos \omega t)\hat{x} + (b \sin \omega t)\hat{y} \text{ m}$$

in which a, b and ω are positive constants ($a > b$) and the time, t is measured in seconds. (a) Show that the orbit of the particle is an ellipse and calculate its equation in terms of x and y. (b) Show that the force which acts upon the particle is always directed towards the origin. (c) Calculate the work done by the force acting on the particle as it moves once around the ellipse. *Hint*: In Cartesian co-ordinates the non-parametric equation of an ellipse is $(x^2/p) + (y^2/q) = 1$ in which p and q are positive quantities.

35. Calculate the kinetic energy of an object of which the mass is 10 kg and the velocity $\bar{v} = 3\hat{x} - 5\hat{y} + 4\hat{z}$ m s^{-1}.

36. A mass of 0,15 kg swings from the lower end of a light cord. Its length is 4 m and the top end is fixed. Calculate the maximum kinetic energy if the angular amplitude of the swing is 8°.

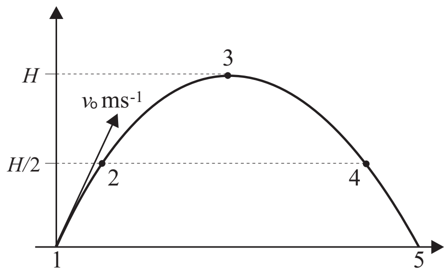
37. Calculate the average power of a waterfall over which water flows at a rate of 11 m^3s^{-1}. The average height of the fall is 15 m and the density of water is 10^3 kg m^{-3}. $g = 9,8$ m s^{-2}.

38. A diver runs at 2 m s^{-1} from a diving board which is 4 m above the surface of the water and makes a belly-flop. The centre of mass of the diver is 1 m above the floor when he stands erect. At what (a) speed, (b) velocity does he strike the surface of the water? Take $\hat{x}$ horizontally forwards and $\hat{y}$ vertically upwards.

39. Electrical energy costs 12,5 c per kWh. Calculate the cost to (a) use an electric lamp which is rated at 150 W for 24 h, (b) use a 2 kW electric heater for 30 days at 6 h per day, (c) bake a cake for 30 minutes in a 4 kW oven, (d) convert 0,5 kg water at 20°C to steam at 100°C. The specific heat capacity of water is $4,117 \times 10^3$ J kg K^{-1} and the specific heat of vaporisation of water is $2,256 \times 10^6$ J kg^{-1}.

40. Assume that a hoisting system which is driven by an electric motor is only 75% efficient. This means that 25% of the energy is lost through dissipative processes. Calculate the height to which such a system would lift a mass of 500 kg at constant velocity against gravity, using a total of 1 kWh of electrical energy. $g = 10 \text{ m s}^{-2}$.

41.



The sketch shows the trajectory of a projectile of which the mass is m kg while it moves under the action of gravity. Disregard frictional drag. The initial speed is v_0 m s^{-1} and the maximum height which it reaches is H metre. The magnitude of gravitational acceleration is g m s^{-2}. At positions 2 and 4, the height is $\frac{1}{2}H$. Take the potential energy as zero at position 1. (N.B. The angle of elevation is not given and should not appear in answers.)
(a) What is the total energy of the projectile at position 1? (b) What is its total energy at position 3? (c) What is its kinetic energy at position 3? (d) What is its speed at position 3? (e) What is the kinetic energy at position 4? (f) What is the speed at position 4? (g) At what angle does it strike the ground?

42. A particle with a mass of 2 kg moves in the following force field:

$$\bar{F} = (3t^2 - 2t)\hat{x} + (2t - 6)\hat{y} + (4t - 6t^2)\hat{z} \quad \text{N}$$

in which the time, t, is measured in seconds. Calculate the change in momentum between $t = 1$ s and $t = 2$s. What is the impulse of the force during this interval of time?

43. A body of mass 10 kg slides on a smooth surface at a speed of 50 m s^{-1}. It explodes and breaks into two portions of 9,0 kg and 1,0 kg respectively. After the explosion the 1 kg portion is at rest. What is the speed of the other portion? What is the Q-value of the process?

44. A mass of 10 kg slides on a smooth horizontal surface at a velocity of $20\hat{x}$ m s^{-1}. It explodes, forming two portions of 2 kg and 8 kg respectively. After the explosion the 2 kg has as velocity of $-16\hat{x}$ m s^{-1}. Calculate the velocity of the other portion. What is the Q-value of the process?

45. A particle with speed 10 m s^{-1} collides with one of similar mass which is at rest. After the collision the egression angles are 25° and -65° respectively. Calculate the final speeds of the two particles. Calculate the Q-value and determine the nature of the interaction (i.e. elastic, endoergic or exoergic).

46. A mass of 3 kg moves at velocity $4\hat{r}$ m s^{-1} when it is involved in a head-on collision with a mass of 8 kg which is moving at $-1,5\hat{r}$ m s^{-1}. The collision is completely inelastic. Calculate the final velocities of the two masses and also the Q-value of the interaction.

47. Transform to a frame of reference in which the 3 kg mass in problem 46 is at rest. What is the initial velocity of the 8 kg mass in this frame of reference? Calculate the final velocities after the collision and also the Q-value in the new frame of reference.

48.Calculate the final velocities of the two masses in problem 46 if the collision had been perfectly elastic.

49. A body with mass 0,3 kg moves at velocity $0,5\hat{r}$ m s^{-1} and collides head-on with one of mass 0,2 kg and velocity $-\hat{r}$ m s^{-1}. The collision is completely inelastic. Calculate the final velocities and also calculate the energy conversion during the collision.

50. Calculate the final velocities of the bodies in problem 49 if the collision were perfectly elastic.

51. Two bodies with masses 2 kg each are involved in an elastic collision. Body A has an initial speed of 2 m s^{-1} and body B is initially at rest. After the collision the egression angle of A is 45° away from its initial direction. Calculate the speeds of both bodies after the collision and also the egression angle of B.

52. Body A has a mass of 1 kg and a velocity of $4\hat{x}$ m s^{-1} when it collides elastically with body B which is at rest and has a mass of 2 kg. After the collision B moves in direction $\hat{r} = (0,8660)\hat{x} + (0,5000)\hat{y}$. Calculate the velocities of both bodies after the collision.

53. A body is involved in a perfectly elastic collision with another of equal mass which is initially at rest. It is an oblique collision. Show that the final velocities are always at right angles.

54. An object with mass 2 kg moves in a time-dependent force field which is given by:

$$\bar{F} = (24t^2)\hat{x} + (36t - 16)\hat{y} + (-12t)\hat{z} \quad \text{N}$$

in which the time, t, is measured in seconds. The initial velocity of the object is $\bar{v}(0) = 6\hat{x} + 15\hat{y} - 8\hat{z}$ m s^{-1}. Calculate (a) the velocity as a function of time, (b) the kinetic energy of the object at $t = 1$ s and $t = 2$ s, (c) the work, $\int \bar{F} \cdot d\bar{r}$, done by the force field on the object between $t = 1$ s and $t = 2$ s. (d) What conclusion may be drawn when answers (b) and (c) are considered?

55. At a given instant a body with mass 10 kg moves along a space curve

at velocity $4\hat{x} - 16\hat{y}$ m s^{-1} and at a later stage at a velocity $8\hat{x} - 20\hat{y}$ m s^{-1}. Calculate the work done on the particle to change its velocity.

56. A mass of 2 kg is suspended from a helical spring with force constant 200 N m^{-1}. From the equilibrium position the mass is displaced to 0,40 m below it and then given a velocity of 3 m s^{-1} downwards. It is allowed to oscillate. (a) Use the principle of energy conservation to calculate the amplitude of the oscillation. (b) Calculate the speed of the mass as a function of its deviation from the equilibrium position. Disregard friction and the mass of the spring.

57. A mass of 2 kg moves on a rough horizontal surface. The coefficient of kinetic friction between the body and the surface is 0,25. $g = 10$ m s^{-2}. The mass collides with the free end of a helical spring with force constant 4 N m^{-1} and compresses the spring through 0,5 m before it comes to rest. (The set-up is identical to that shown in figure 2.5-7) Calculate the speed of the mass when it makes contact with the spring.

58. A block of wood with mass 9,98 kg is suspended from four parallel light cords in such a way that it remains horizontal when disturbed from equilibrium. A bullet of mass 0,02 kg penetrates the block and is stopped in it. The block with the bullet in it swings to a maximum height of 0,1 m vertically above the position where the block was at rest. Calculate the speed with which the bullet struck the block. Calculate the Q-value of the interaction. $g = 10$ m s^{-2}. The set-up is identical to that shown in figure 2.6-3.

59. In a similar set-up as that in problem 58, the block has a mass of 1 kg and the bullet, 2 g. The bullet strikes the block at 500 m s^{-1}, passes through it and exits at 100 m s^{-1}. Assume that no slivers break away from the block of wood. How high will the block rise when swinging away from equilibrium? What is the Q-value of the interaction?

60. When a scale pan of mass 0,300 kg is suspended from a light helical spring, an extension of 150 mm occurs. A ball of soft Plasticine with mass 0,200 kg is at rest at a height of 200 mm above the scale pan when it is dropped. $g = 10$ m s^{-2}. How far will the scale pan with the Plasticine in it move downwards before coming to rest and reversing its motion? Assume that the interaction between the scale pan and the Plasticine is completely inelastic.

61. Calculate the Q-value of the reaction ^{7}Li(p, α)α. The mass of an α-particle (i.e.^{4}He) is 4,0026036 u, that of ^{7}Li is 7,0160053 u and that of a proton (i.e. ^{1}H), 1,0078252 u. Compare this with the information given in example problem 2.7.2.

62. Use the sketch in figure 2.7-1. A proton with energy 5 MeV, strikes a ^{11}B-nucleus which is at rest. In the resulting nuclear reaction an α-particle and a

^{8}Be-nucleus are formed. The α-particle is detected in detector D at right angles. Calculate the energies of the product particles and the egression angle of the ^{8}Be-nucleus. The Q-value of the reaction is 8,588 MeV.

63. In the enormous gravitational field of the sun and the existing high temperature at its centre, hydrogen is being converted to helium. The name of the process is **nuclear fusion** which converts four protons to one nucleus of ^{4}He. The rest mass of a proton is 1,0078252 u, and that of ^{4}He, 4,0026036 u. Calculate the energy output of one such fusion reaction in MeV.

64. When a ^{235}U-nucleus captures a slow neutron, it forms the highly unstable nucleus ^{236}U which immediately breaks up and forms two lighter nuclei and may also emit neutrons. The name of the process is **nuclear fission**. It may be represented as follows:

$$^{235}\mathrm{U} + \mathrm{n} \quad \rightarrow \quad ^{236}\mathrm{U} \quad \rightarrow \quad \mathrm{A} + \mathrm{B} + \nu\mathrm{n}$$

in which $\nu = 0,\ 1,\ 2,\ 3\ \ldots =$ number of neutrons that are emitted. The value of ν depends on which fission nuclei, A and B, are formed. The reaction does not occur in a unique way and the average value of ν is 2,5 neutrons per fission. Calculate the number of neutrons emitted and the release of energy, in the case where the fission nuclei are ^{141}Ba and ^{92}Kr. The masses of the involved particles are as follows: ^{235}U: 235,043933 u, neutron: 1,008665 u, ^{141}Ba: 140,91374 u, ^{92}Kr: 91,88490 u.

65. A laser beam with power 100 W strikes a mirror at right angles and is totally reflected. The cross-sectional area of the beam is 1 mm^2. Calculate the force exerted on the mirror and also the radiation pressure. Repeat the problem for an incident angle of 45°. (Compare this problem with problem 12.)

66. The rest mass of an electron is $9,108 \times 10^{-31}$ kg and its electric charge, $-1,602 \times 10^{-19}$ C. Through what electric potential difference must an electron be accelerated to attain a speed of 0,999999c relative to the laboratory. $c =$ speed of light in free space.

67. When a fast electron interacts with a metal to come to rest, a portion of its energy may be converted to X-rays by the process known as **Bremsstrahlung** (German for **breaking radiation**). Calculate the maximum possible X-ray energy which can be produced by an electron of which the total energy is 10 MeV. The rest mass of an electron is given in the previous problem.

68. At what speed must a particle move in order that its rest energy is 0,1 of its total energy?

Chapter 3

ROTATION AND THE DYNAMICS OF BODIES

In the previous chapter, the motion of mainly *point masses* was considered. Where an actual *body* was considered, its spatial qualities did not influence the problem in question. In this chapter a number of special cases of the motion of rigid bodies, which cannot be represented by point masses, will be studied.

3.1 The centre of mass of a rigid body

A **rigid body** may be thought of as a collection of point masses which retain their positions relative to each other. Although no bodies exist which conform strictly to this definition, many are good approximations if the conditions are favourable. A wheel, for example, will behave according to this ideal model if the rate at which it rotates or the forces which act on it, are not large enough to distort its shape.

In chapter 2 reference was quite often made to the **centre of mass** of a body and it was mainly used to indicate the position of a body which was represented by a point mass. In this sense the centre of mass was thought of as a point (i.e. a position) at which all the mass of the body was concentrated. Until now, the calculation of this position was not important. In chapter 3, however, it plays an important role and its definition and calculation will have to be treated in detail.

3.1.1 The centre of mass of a collection of discrete point masses

When one speaks of **discrete** point masses, it means that they are *distinguishable* and *countable*. Many examples exist where bodies may be represented by a collection of discrete point masses.

The **centre of mass** of a collection of discrete point masses is defined as follows:

$$\bar{R} = (\sum_i m_i \bar{r}_i)/(\sum_i m_i) = (\sum_i m_i \bar{r}_i)/M \qquad 3.1(1)$$

in which $\bar{r}_i$ is the position vector of particle number i of which the mass is m_i. M is the mass of the entire collection of point masses. The summations have to be calculated by taking all the point masses into account.

Examples:

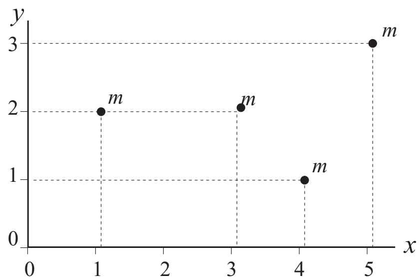
1. 

Figure 3.1-1

Figure 3.1-1 shows a collection of point masses. The mass of each is m kilogram. Their positions are given on the sketch. The units along the axes are metres. Calculate the centre of mass of this collection.

In this two-dimensional problem we have:

$$\begin{aligned} \bar{R} &= X\hat{x} + Y\hat{y} \\ X &= \frac{m + 3m + 4m + 5m}{m + m + m + m} = \frac{13}{4} = 3,25 \quad \text{m} \\ Y &= \frac{m + 2m + 2m + 3m}{m + m + m + m} = \frac{8}{4} = 2,00 \quad \text{m} \end{aligned}$$

This calculation could have been made in a single step as follows:

$$\begin{aligned} \bar{R} &= \frac{m(4\hat{x} + \hat{y}) + m(\hat{x} + 2\hat{y}) + m(3\hat{x} + 2\hat{y}) + 3(5\hat{x} + 3\hat{y})}{4m} \\ &= \frac{13m\hat{x} + 8m\hat{y}}{4m} = 3,25\hat{x} + 2,00\hat{y} \quad \text{m} \end{aligned}$$

2. A hypothetical molecule consists of the following atoms: 4 atomic mass units (u) at position $\bar{r}_4 = 3\times10^{-10}\hat{z}$ m $=$ 300 picometre (pm), 2 u at position $\bar{r}_2 = 300\hat{y}$ pm, 3 u at position $\bar{r}_3 = 300\hat{x}$ pm and 6 u at the origin. Calculate the centre of mass of the molecule.

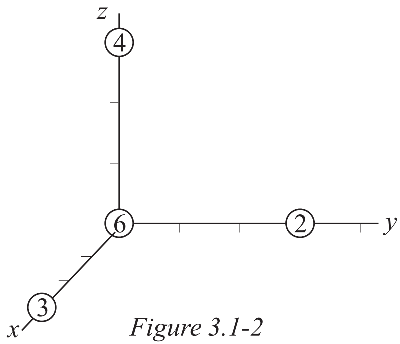


Figure 3.1-2

A representation of this molecule is shown in figure 3.1-2. The units along the axes are picometre (pm).

$$\begin{aligned}\bar{R} &= \frac{900\hat{x} + 600\hat{y} + 1200\hat{z}}{15} \quad \text{pm} \\ &= 60\hat{x} + 40\hat{y} + 80\hat{z} \quad \text{pm} \\ &= (6\hat{x} + 4\hat{y} + 8\hat{z}) \times 10^{-11} \quad \text{metre}\end{aligned}$$

3.1.2 The centre of mass of a continuous mass distribution

Matter consists of atoms of which almost all the mass is concentrated in the nuclei which, with a high degree of accuracy, may be treated as point masses. In principle the centre of mass of any material object may thus be calculated by means of equation 3.1(1). Even the smallest particle that can be observed by the naked eye contains so many atoms that such a calculation would be totally impossible. Because atoms are so extremely small, a macroscopic model, which works very well, may be constructed. In stead of taking the atomic structure of a body into account, the mass of each molecule is considered to be spread out over the entire portion of the volume that it represents. The motivation for this assumption is that all matter seems to be continuous when viewed macroscopically.

With this model as basis, the following macroscopic property of matter may be defined: The **volumetric mass density** or **bulk mass density** (*density* in short) of the material of which a body consists, is its mass per unit volume.

$$\rho = \frac{dm}{dV} \qquad 3.1(2)$$

in which dm is the mass allocated to the element of volume dV in accordance with the model described above. In general the density of a body may be a scalar field (scalar function of position) which may have different values at different positions. If a body consists of a material of which the density is the same at all positions, it is said to be **homogeneous**.

The **average density**, ρ_{av}, of a body, is its total mass divided by its volume.

$$\rho_{av} = m/V \qquad 3.1(3)$$

If a body is homogeneous, its density at each position is equal to the average density of the whole body. The SI units of density are $\mathrm{kg\,m^{-3}}$ and $[\rho] = [\mathrm{M\,L^{-3}}]$.

It is of interest to note that the original definition of the kilogram was such that the density of pure water at 4°C (277,15 K) and normal pressure is 10^3 $\mathrm{kg\,m^{-3}}$. The density of atomic nuclei is in the order of magnitude 10^{17} $\mathrm{kg\,m^{-3}}$ (100 000 tons per cubic millimetre). This enormous density exemplifies the extreme emptiness of matter when it is viewed macroscopically.

When material is in sheet form, it is often convenient to work with its **surface mass density** (*surface density* in short), α.

$$\alpha = \frac{dm}{dA} \qquad 3.1(4)$$

in which dm is the mass of an element of area, dA, at the position where α is measured.

The average surface density, α_{av}, is the total mass of a sheet divided by its area.

$$\alpha_{av} = \frac{m}{A} \qquad 3.1(5)$$

The surface density at all positions on a uniform sheet (the thickness is the same at all positions) is equal to its average surface density. The SI units of surface density are $\mathrm{kg\,m^{-2}}$ and $[\alpha] = [\mathrm{M\,L^{-2}}]$.

Comment: In the paper industry, different thicknesses of paper are distinguished by the specification of their surface densities. The units of *gram per square metre* usually abbreviated by *gsm* are used for this purpose. This book is printed on 40 gsm paper.

For material in the shape of a rod, its mass per unit length is used. This is known as its **linear mass density** or **linear density**, μ.

$$\mu = \frac{dm}{dL} \qquad 3.1(6)$$

in which dm is the mass of an element of length, dL, at the position where μ is measured.

The average linear density, μ_{av}, is the total mass of a rod divided by its length.

$$\mu_{av} = \frac{m}{L} \qquad 3.1(7)$$

The linear density at each position on a uniform homogeneous rod, is the same as its average linear density. The SI units of linear density are $\mathrm{kg\,m^{-1}}$ and $[\mu] = [\mathrm{M\,L^{-1}}]$.

For the determination of the centre of mass of a body, it can, in principle, be subdivided into a large number of mass elements and this allows the application of definition 3.1(1).

$$\bar{R} \approx \frac{\sum_i \Delta m_i \bar{r}_i}{\sum_i \Delta m_i}$$

in which Δm_i is the mass of element number i at position $\bar{r}_i$. The approximation becomes an equality when the limit of the sum is calculated when the magnitudes of the mass elements tend to zero.

$$\bar{R} = \lim_{\Delta m_i \to 0} \frac{\sum_i \Delta m_i \bar{r}_i}{\sum_i \Delta m_i} = \lim_{\Delta m_i \to 0} \frac{\sum_i \Delta m_i}{M}$$

in which $M = \lim_{\Delta m_i \to 0} (\sum_i \Delta m_i)$ is the mass of the body.

The limits of these two summations are, by definition, integrals and the calculation of the centre of mass may be rewritten as

$$\bar{R} = (\int \bar{r}\, dm)/(\int dm) = \frac{1}{M} \int \bar{r}\, dm \qquad 3.1(8)$$

in which the integrals have to be calculated over the whole body in each case.

The element of mass, dm, will usually be written in terms of μ, α or ρ, and one or more elements of position, depending on whether the body is one, two or three-dimensional.

Examples: 1.

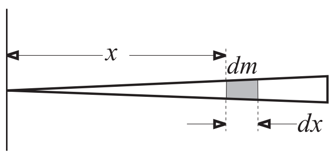


Figure 3.1-3

A thin wedge-shaped rod is 2 m long and has a linear density which depends on x, the distance from the thin edge. It is given by the function $\mu = 0,05x$ kg m^{-1}. Calculate the position of the centre of mass.

Consider an element of length, dx, which is x m from the thin edge. Its mass is given by

$$dm = \mu dx = 0,05x\, dx$$

Consideration of the symmetry shows that the centre of mass must lie on the rod. If the rod is placed along the x-axis as shown in figure 3.1-3, the position has an x- component only. The calculation is made by using equation 3.1(8):

$$\begin{aligned} X &= (\int_0^2 x dm)/(\int_0^2 dm) \\ &= (\int_0^2 0,05x^2 dx)/(\int_0^2 0,05x dx) = 1,333 \quad \text{m} \end{aligned}$$

2.

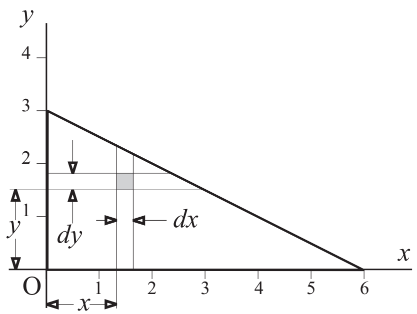


Figure 3.1-4

A uniform homogeneous flat steel sheet is to be used in the construction of a ship. It is in the shape of a rectangular triangle of which the sides adjacent to the right angle have lengths of 3 m and 6 m respectively.

Because the sheet is uniform and homogeneous, its surface density, α, is the same at all positions. Choose a frame of reference as shown in Figure 3.1-4. Consider an element of area of magnitude $dx\,dy$ at position $\bar{r} = x\hat{x} + y\hat{y}$ metre. The mass of this element is

$$dm = \alpha dx\, dy$$

The calculation of the total mass of this plate requires two integrations, firstly to x to calculate the mass of a strip parallel to the x-axis and then to y to cover the entire plate. The Cartesian equation of the hypotenuse is $y = -0,5x + 3$ and from this follows that $x = 6 - 2y$ along this line. The mass is given by

$$\begin{aligned} m &= \int_0^3 (\int_0^{6-2y} \alpha dx) dy = \int_0^3 (\alpha x|_0^{6-2y}) dy = \alpha \int_0^3 (6 - 2y) dy \\ &= \alpha(6y - y^2|_0^3) = 9\alpha \text{ kg} \end{aligned}$$

Comment: The mass could have been calculated directly by $m = \text{area}\times\alpha = \frac{1}{2}\text{base}\times\text{height} = \frac{1}{2} \times 6 \times 3 \times \alpha = 9\alpha$ kg, which would have been much easier. This **surface integral** was used as an illustration of the technique which will be required for the completion of the calculation.

The calculation of the position of the centre of mass by use of Equation 3.1(8), requires a further two integrations in order to cover the entire body.

$$\begin{aligned} \bar{R} &= \left(\frac{1}{M} \int \bar{r} dm \right)_{\text{over the entire body}} \\ &= \left(\frac{1}{9\alpha} \int (x\hat{x} + y\hat{y}) \alpha dx\, dy \right)_{\text{over the entire body}} \\ &= \frac{1}{9} \int_0^3 [\int_0^{6-2y} (x\hat{x} + y\hat{y}) dx] dy \end{aligned}$$

When the integration to x is performed, the calculation is made for a strip parallel to the x-axis and then y is taken as constant.

$$\bar{R} = \frac{1}{9} \int_0^3 [(\frac{1}{2}x^2)\hat{x} + (xy)\hat{y}|_0^{6-2y}] dy$$

$$
\begin{aligned}
&= \frac{1}{9}\int_0^3 [(18 - 12y + 2y^2)\hat{x} + (6y - 2y^2)\hat{y}]dy \\
&= \frac{1}{9}[(18y - 6y^2 + \frac{2}{3}y^3)\hat{x} + (3y^2 - \frac{2}{3}y^3)\hat{y}|_0^3] \\
&= 2\hat{x} + \hat{y} \text{ m}
\end{aligned}
$$

3. Calculate the centre of mass of a solid homogeneous right circular cone with base radius R metre and height H metre.

The fact that the cone is homogeneous means that its density, ρ, is constant at each position in it. In principle a calculation similar to that in the previous problem may be made in three dimensions. Using Cartesian co-ordinates, this would be a fairly difficult calculation. Cylindrical polar co-ordinates would be more suitable for a body which has axial symmetry.

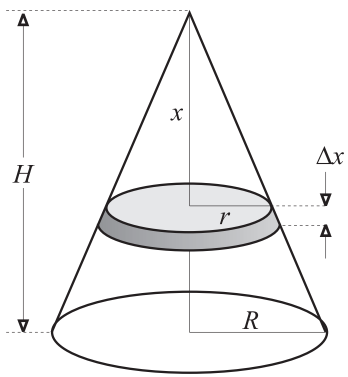


Figure 3.1-5

The symmetry of the cone, however, allows the reduction of the problem to one dimension. Since it is symmetrical about its vertical axis (see Figure 3.1-5), the centre of mass must lie on this axis.

Consider a circular slice of the cone with radius r and thickness dx at right angles to the axis and at distance x from the apex. Because its thickness is infinitesimal, its volume may be approximated by that of a solid cylinder.

$$dV = \pi r^2 dx$$

so that its mass will be

$$dm = \rho\pi r^2 dx$$

But r and x are not independent and one has to be written in terms of the other. From the similar cones in Figure 3.1-5 it may be seen that $r = (R/H)x$ so that the mass of the element may be written as

$$dm = \rho\pi(R^2/H^2)x^2 dx$$

The position of the centre of mass of the cone is given by

$$X = [(\int x\, dm)/(\int dm)]_{\text{over the entire body}}$$

$$= [\int_0^H \rho\pi(R^2/H^2)x^3dx]/[\int_0^H \rho\pi(R^2/H^2)dx]$$
$$= (\frac{1}{4}x^4|_0^H)/(\frac{1}{3}x^3|_0^H) = 0,75H \text{ metre}$$

The centre of mass of the cone is on its vertical axis at distance 0,25H from the circular base.

3.2 Rotational kinematics

Consider any object relative to a suitable *inertial frame of reference*. If each point in the body has the same acceleration and velocity, it is said that the body experiences a **pure translational motion**. A **pure rotational motion**, is one in which all points along one straight line remain at rest while all other points move in circles of which the centres are on this line which is known as the **axis**.

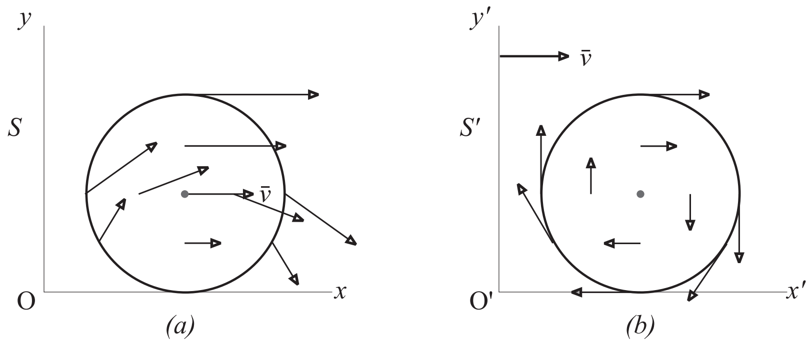


Figure 3.2-1

The motion of a body may be a combination of these two kinds of motion. A wheel which rolls over a plane surface without slipping is an example of combined translation and rotation. Figure 3.2-1(a) shows the velocity vectors of a number of points on a rigid wheel relative to inertial system S in which the plane over which the wheel is rolling, is at rest. The axis of the wheel moves at velocity $\bar{v} = v\hat{x}$, the uppermost point at velocity $\bar{v} = 2v\hat{x}$ whilst the point in contact with the plane is always at rest. Figure 3.2-1(b) shows the same wheel relative to S' in which the axis is at rest. The velocity of each point on the wheel may be calculated by means of the Galilei transformation. In frame S', in which the wheel executes a pure rotational motion, each point on the perimeter has the same speed v. This example illustrates a general principle that may be proved mathematically: *The most general motion that a body can execute, may be described as the combination of translation of its centre of mass and rotation about its centre of mass.* A general description of such a motion requires knowledge of a field of mathematics which lies outside the scope of this book.

3.2.1 Rotation angle, angular velocity and angular acceleration

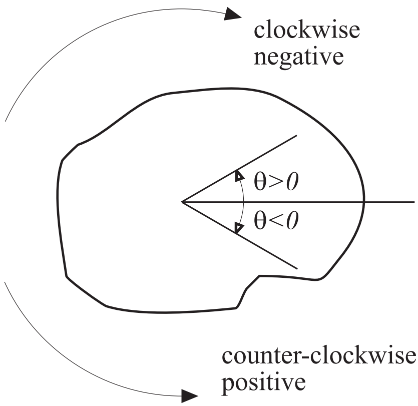


Figure 3.2-2

When a body is limited to rotation about a stationary axis, O, which is perpendicular to the plane of the diagram, it possesses one degree of freedom only, the **rotation angle**, θ. If the body rotates **counter-clockwise** (also called **anticlockwise**), the sign of θ is taken to be positive. If the rotation is **clockwise**, the sign is negative. This is in accordance to the sign convention for angles used in mathematics. An observer viewing the rotating body from the opposite side of the axis would assign the opposite designations. The terms are explained in Figure 3.2-2.

For the description of the rotation of a body, a straight line which is at rest relative to a suitable inertial frame of reference is chosen through the axis of rotation. The angle between this line and the position vector of any point on a rigid body is sufficient to describe its **orientation** (or **"angular position"**). It is customary (though not necessary) to choose the numerical value of this angle as zero when $t = 0$. With this choice, the value of θ gives the rotation of the body from instant $t = 0$. A positive value of θ indicates that the body has a net rotation in a counter-clockwise sense and a negative value, a net rotation in a clockwise sense. If θ is zero, it could indicate that the body had rotated through equal angles clockwise and counter-clockwise or that it had not rotated at all.

If θ is known as a function of time, the orientation is known at each instant from $t = 0$. The SI units in which this angle is measured, are *radians*. In practice, revolutions and degrees are also used, and the reader should be skilled in the conversion of these units to radians. The use of radians has many advantages. In this text, it will be assumed that the angle of rotation is measured in radians unless it is otherwise specified. Angles are dimensionless.

The **angular velocity** is defined as the rate at which the angle of rotation changes. It is indicated by ω or $\dot{\theta}$.

$$\omega = \dot{\theta} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad 3.2(1)$$

The SI units of angular velocity are $\mathrm{rad\,s^{-1}}$, and $[\omega] = [\mathrm{T}^{-1}]$. Angular velocity is measured by means of a **tachometer**. Tachometers which measure the angular velocity of motor-car engines, usually measure in revolutions per minute. The conversion of revolutions per minute to radians per second is

$$(\mathrm{rad\,s^{-1}}) = (\text{revolutions per minute}) \times \frac{2\pi}{60}$$

If the angular velocity of a rotating body changes, it is said to experience an **angular acceleration** which is defined as the rate at which the angular velocity changes. Angular acceleration is indicated by α, $\dot{\omega}$ or $\ddot{\theta}$.

$$\alpha = \dot{\omega} = \ddot{\theta} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \qquad 3.2(2)$$

The SI units of angular acceleration are $\mathrm{rad\,s^{-2}}$ and $[\alpha] = [\mathrm{T}^{-2}]$.

3.2.2 The relationship between the translation and rotation of a point mass

A point mass moves along a fixed circle with radius r metre. Choose a frame of reference with origin at the centre of the circle. The velocity vector is always tangential to the circle. Whether the speed of the point remains constant or not, it accelerates because the direction of the velocity vector changes continuously. This acceleration implies the existence of a **constraining force** which keeps the point mass in its circular orbit. This force is called a **centripetal force** and will be studied in detail in section 3.3. It will be shown that this force is always directed towards the centre of the circle and that it cannot influence the speed of the body. If the speed of the body changes, it will be the result of a component of the acceleration which is *tangential* to the circular orbit.

If the point mass moves a distance s along the circular orbit, its position vector describes an angle θ as is shown in Figure 3.2-3. By definition

$$\theta = \frac{s}{r} \qquad 3.2(3)$$

and

$$s = r \times \theta \qquad 3.2(4)$$

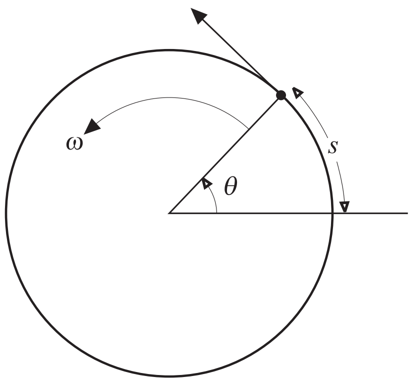


Figure 3.2-3

To calculate the relationship between the angular velocity and the translation speed, equation 3.2(3) is differentiated to the time, t.

$$\omega = \frac{d\theta}{dt} = \frac{1}{r}\frac{ds}{dt} = \frac{v}{r} \qquad 3.2(5)$$

$$\text{and} \quad v = r \times \omega \qquad 3.2(6)$$

The relationship between the angular acceleration and the *tangential* component of the translational acceleration follows directly if Equation 3.2(5) is differentiated to the time.

$$\alpha = \frac{d\omega}{dt} = \frac{1}{r}\frac{dv}{dt} = a_s/r \qquad 3.2(7)$$

$$\text{and} \quad a_s = r \times \alpha \qquad 3.2(8)$$

Since all the particles in a body which rotates about a fixed axis move on circular orbits, these relationships apply to their motion. When the body is rigid, each particle must have the same angular velocity and the same angular acceleration. All particles in such a body do not have the same speed (see Figure 3.2-1). Positions on circles with equal radii will, however, have the same speed. This follows directly form Equations 3.2(5) and 3.2(6). The speed of a given particle in a rotating body will be proportional to its distance from the axis, i.e. the radius of the circle along which it moves.

3.2.3 The solution of problems on rotational kinematics

If $\theta = \theta(t)$ is known, the angular velocity and the angular acceleration may be calculated by differentiating it to time once and twice respectively. By means of these three functions (the angle, angular velocity and angular acceleration) any kinematic information regarding the system may be calculated.

If the angular velocity, ω, is known as a function of time, the angular acceleration may be calculated directly by differentiation to time. The angle θ can also be calculated. Since $\omega = d\theta/dt$, it follows that

$$\theta = \int \omega \, dt \qquad 3.2(9)$$

The integration constant may be calculated if the value of θ is known at a given instant. If no value of θ is known at an instant, the amount of rotation between two given instants may be calculated by means of a definite integral.

$$\theta_2 - \theta_1 = \Delta\theta = \int_{t_1}^{t_2} \omega \, dt \qquad 3.2(10)$$

The value of this integral may be represented by an area similar to that shown in figure 1.5-1. The meaning of positive, negative and zero answers has been explained in 3.2-1.

If the angular acceleration, α, is known as a function of time, the angular velocity may be calculated by integration. Since $\alpha = d\omega/dt$, it follows that

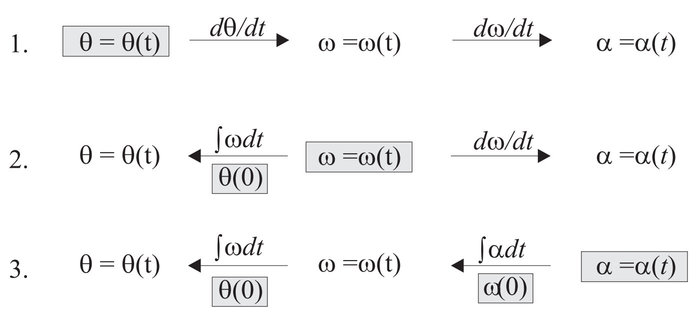
$$\omega = \int \alpha \, dt \qquad 3.2(11)$$

from which the angular velocity may be calculated as a function of time if its numerical value at one given instant is known. If no numerical value of ω at any instant is known, the change in the angular velocity between two given instants may be calculated by means of a definite integral.

$$\omega_2 - \omega_2 = \Delta\omega = \int_{t_1}^{t_2} \alpha \, dt \qquad 3.2(12)$$

This change in angular velocity may be represented by an area similar to that in Figure 1.5-2.

The following is a schematic representation of the solution of rotational kinematic problems in which time is the independent variable:



In some problems it is more convenient to use the rotation angle rather than time as independent variable. The way in which the angle is introduced and time eliminated, is identical to the method used for translational kinematics in section 1.6.

$$
\begin{aligned}
\text{By definition} \qquad \alpha &= \frac{d\omega}{dt} \\
&= \frac{d\omega}{d\theta}\frac{d\theta}{dt} \qquad \text{(chain rule)} \\
&= \frac{d\omega}{d\theta}\omega \qquad (\omega = d\theta/dt) \\
\text{so that} \qquad \alpha\, d\theta &= \omega d\omega
\end{aligned}
$$

If $\omega = \omega_1$ when $\theta = \theta_1$ and $\omega = \omega_2$ when $\theta = \theta_2$, it follows that

$$\int_{\theta_1}^{\theta_2} \alpha\, d\theta = \int_{\omega_1}^{\omega_2} \omega\, d\omega$$

in which any one of ω_1, ω_2, θ_1, θ_2 or α may be unknown, or the integral may be used to calculate $\omega = \omega(\theta)$. These problems are an exact analogue of the translation problems which were discussed in section 1.6.

Examples:

1. The average radius of the earth is $6,371 \times 10^6$ m. Calculate the angular velocity of the earth in radians per second. Also calculate the speed of a point on the equator in $\mathrm{m\,s^{-1}}$ and $\mathrm{km\,h^{-1}}$. As a result of its motion around the sun, the earth completes more than one rotation in one solar day (24 hours = $8,640 \times 10^4$s). A complete revolution takes one sidereal day which is equal to $8,616 \times 10^4$ s.

$$
\begin{aligned}
\omega &= \frac{2\pi}{8,616 \times 10^4} = 7,293 \times 10^{-5}\ \mathrm{rad\,s^{-1}} \\
v &= r \times \omega = 6,371 \times 10^6\ \times\ 7,293 \times 10^{-5} = 4,646 \times 10^2\ \mathrm{m\,s^{-1}} \\
&= (4,646 \times 10^2) \times 10^{-3} \times 3600 = 1,673 \times 10^3\ \mathrm{km\,h^{-1}}
\end{aligned}
$$

2. An electric motor initially rotates at 2400 revolutions per minute and is then slowed down at a constant angular acceleration to attain an angular velocity of 1800 revolutions per minute in 8 seconds. (a) Calculate its angular acceleration. (b) Calculate the number of revolutions completed during the 8 seconds when it was slowed down.

(a) $\omega_1 = \frac{2400\times 2\pi}{60} = 80\pi\ \mathrm{rad\,s^{-1}}$ and $\omega_2 = \frac{1800\times 2\pi}{60} = 60\pi\ \mathrm{rad\,s^{-1}}$

$$\Delta\omega = \omega_2 - \omega_1 = \int_0^t \alpha\, dt \quad \text{so that} \quad 60\pi - 80\pi = \int_0^8 \alpha\, dt = 8\alpha$$

and $\alpha = -2,5\pi\ \mathrm{rad\,s^{-2}}$ (b) Since

$$\begin{aligned} \alpha\, d\theta &= \omega\, d\omega \\ \int_0^\theta \alpha\, d\theta &= \int_{80\pi}^{60\pi} \omega\, d\omega \\ -2,5\pi\theta &= \frac{1}{2}(60\pi)^2 - \frac{1}{2}(80\pi)^2 \\ \theta &= 560\pi \text{ rad} \\ &= 560\pi/2\pi = 280 \text{ revolutions} \end{aligned}$$

3. A wheel rotates about a fixed axis AB. Take counter-clockwise positive as observed from B. The angular acceleration of the wheel is 4 $\text{rad}\,\text{s}^{-2}$. The initial angular velocity is given by $\omega(0) = 8$ $\text{rad}\,\text{s}^{-1}$ as observed from B. Take $\theta(0) = 0$. (a) Calculate the angular velocity as a function of time. (b) Calculate the rotation angle as a function of time. (c) When is the wheel at rest? (d) Through what angle does it rotate before coming to rest? (e) When is $\theta = 0$?

(a) Since $\alpha = d\omega/dt$, it follows that $d\omega = \alpha dt$ and

$$\begin{aligned} \int_{-8}^{\omega} d\omega &= \int_0^t 4\, dt \\ \omega|_{-8}^{\omega} &= 4t|_0^t \\ \text{so that} \qquad \omega &= -8 + 4t \text{ rad}\,\text{s}^{-1} \end{aligned}$$

(b) Since $\omega = d\theta/dt$, we have that $d\theta = \omega dt$ and

$$\int_0^\theta d\theta = \int_0^t (-8 + 4t)dt \qquad \text{so that} \qquad \theta = -8t + 2t^2 \text{ rad}$$

(c)The wheel is at rest when $\omega = 0$, i.e. when $-8 + 4t = 0$ or $t = 2$ s.

(d) First method: From answers (b) and (c) it follows that

$$\theta(2) = -8(2) + 2(2)^2 = -8 \text{ rad}$$

This means that the wheel turns through 8 rad clockwise from time $t = 0$ until it stops.

Second method:

$$\begin{aligned} \int_0^\theta \alpha\, d\theta &= \int_{-8}^{0} \omega d\omega \\ 4\theta|_0^\theta &= \frac{1}{2}\omega^2|_{-8}^0 \qquad (\alpha = 4 \text{ rad}\,\text{s}^{-2}) \\ \text{from which follows} \qquad \theta &= -8 \text{ rad} \end{aligned}$$

(e) When $\theta = 0$, then $2t^2 - 8t = 0$. This gives two values for t: $t = 0$ and $t = 4$ s. The first value of t is the instant when the stop-watch was set in motion and when the measurement of θ commenced. The second value is the instant at which the wheel turned through equal angles clockwise and counter-clockwise.

3.3 Circular motion. Centripetal acceleration

In section 3.2.2 it was seen that a mass which moves on a circular path, constantly accelerates because its velocity changes direction. In section 3.3 this acceleration and the force which causes it, will be studied in detail.

3.3.1 Centripetal acceleration

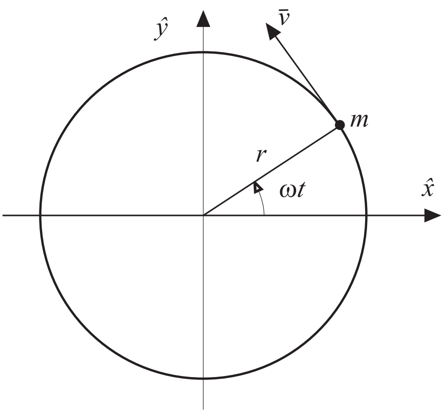


Figure 3.3-1

Consider a body with mass m kilogram which moves at a constant speed of v m s^{-1} along a circular path under the action of a **constraining force**. The magnitude of its position vector remains constant and it rotates at a constant angular velocity of $\omega = v/r$ as it was previously shown for similar circumstances.

Let the radius position vector coincide with the x-axis at instant $t = 0$. At any later instant, the angle between the position vector and the x-axis is given by $\theta = \omega \times t$.

In the chosen frame of reference (see Figure 3.3-1) the position vector of the mass is given by

$$\bar{r} = \bar{r}(t) = (r \cos \omega t)\hat{x} + (r \sin \omega t)\hat{y} \qquad 3.3(1)$$

The velocity vector is given by the derivative of the position vector.

$$\bar{v} = d\bar{r}/dt = (-r\omega \sin \omega t)\hat{x} + (r\omega \cos \omega t)\hat{y} \qquad 3.3(2)$$

and the acceleration vector by

$$\bar{a} = d\bar{v}/dt = (-r\omega^2 \cos \omega t)\hat{x} + (-r\omega^2 \sin \omega t)\hat{y} \qquad 3.3(3)$$

From these three functions some some interesting information follows about circular motion.

From 3.3(1) and 3.3(2) follow that

$$\begin{aligned}\bar{v}\bullet\bar{r} &= [(-r\omega\sin\omega t)\hat{x} + (r\omega\cos\omega t)\hat{y}]\bullet[(r\cos\omega t)\hat{x} + (r\sin\omega t)\hat{y}] \\ &= (-r\omega\sin\omega t)(r\cos\omega t) + (r\omega\cos\omega t)(r\sin\omega t) \\ &= 0\end{aligned}$$

which confirms that the position vector and the velocity vector are orthogonal, a fact which was assumed when the motion was specified. The velocity vector of the motion is always tangential to the space curve.

In Equation 3.3(3) a common factor of $-\omega^2$ exists for the terms on the right, and this equation may be rewritten as

$$\bar{a} = -\omega^2[(r\cos\omega t)\hat{x} + (r\sin\omega t)\hat{y}] = -\omega^2\bar{r} \qquad 3.3(4)$$

Since ω^2 is a positive number for all real values of ω, Equation 3.3(4) shows that the direction of the acceleration vector is always opposite to that of the position vector. In the chosen frame of reference, the position vector always points away from the origin which is the centre of the circular path. Therefore the acceleration points towards the centre.

The above calculations were based on the assumption that the speed of the mass is constant. By calculating the speed from the velocity vector, it may be tested if the results are in accordance with this assumption.

$$\begin{aligned}v = |\bar{v}| = (\bar{v}\bullet\bar{v})^{\frac{1}{2}} &= (r^2\omega^2\sin^2\omega t + r^2\omega^2\cos^2\omega t)^{\frac{1}{2}} \\ &= r\omega(\sin^2\omega t + \cos^2\omega t)^{\frac{1}{2}} = r\omega\end{aligned}$$

which is constant since r and the angular velocity, ω, are constant.

The acceleration which was calculated in 3.3(4) does not change the speed of the mass. The constant change in velocity which occurs is the result of a change in direction only and is caused by the **centripetal acceleration** which was calculated. Since this acceleration is always perpendicular to the circular path, it will be indicated by the symbol $\bar{a}_n$.

From the relationship $\omega = v/r$ in equation 3.2(6), the centripetal acceleration may also be expressed in terms of the speed.

$$\bar{a}_n = -\omega^2\bar{r} = -(v^2/r^2)\hat{r} = -(v^2/r)\bar{r} \qquad 3.3(5)$$

This result may be formulated as follows: A body which moves in a circular orbit of radius r metre at a constant speed of v m s^{-1} (or a constant angular velocity ω rad s^{-1}), experiences an acceleration with magnitude $a_n = v^2/r = \omega^2 r$ which is directed towards the centre of the circle.

If the body is to be kept in the same circular orbit at a higher speed, the magnitude of the centripetal acceleration will need to be accordingly larger. For a given radius, the centripetal acceleration is proportional to the square of the speed. On the other hand, the magnitude of the centripetal acceleration will be less if the speed remains the same but the radius of the circular orbit is increased. For a given speed, the magnitude of the centripetal acceleration is inversely proportional to the radius of the circle.

Examples

1. An object which is tied to the end of a light cord, is swung to describe a circular orbit with radius 2 m at a constant angular velocity of one revolution per second. Calculate the speed of the object and the centripetal acceleration.

$$\begin{aligned} \omega &= 1 \text{ revolution per second} = 2\pi \ \mathrm{rad\,s^{-1}} \\ v &= r \times \omega = 2 \times 2\pi = 4\pi \ \mathrm{m\,s^{-1}} \\ a_n &= r \times \omega^2 = 2(2\pi)^2 = 8\pi^2 = 78,96 \ \mathrm{m\,s^{-2}} \\ \text{also} \quad a_n &= v^2/r = (4\pi)^2/2 = 78,96 \ \mathrm{m\,s^{-2}} \end{aligned}$$

2. Consider the same set-up as in the previous problem. Calculate the centripetal acceleration of the object if (a) the speed is the same as in the previous problem, but the radius equal to 4 m, (b) the radius is 2 m but the speed only half of that in problem 1. (a) $a_n = v^2/r = (4\pi)^2/4 = 39,48 \ \mathrm{m\,s^{-2}}$ (b) $a_n = v^2/r = (2\pi)^2/2 = 19,74 \ \mathrm{m\,s^{-2}}$

3.3.2 Centripetal force

In an inertial system the acceleration of a mass can occur only if it is caused by an unbalanced external force. The force which causes a centripetal acceleration is called a **centripetal force**. This designation is somewhat unfortunate in that people might infer that this is a different kind of force which was inadvertently omitted in section 2.2. By means of suitable examples it will be shown that any force may be employed to fulfil the function of centripetal force.

According to Newton's second law of motion, the centripetal force is given by

$$\bar{F} = m\bar{a} = -m\omega^2\bar{r} = -m(v^2/r)\hat{r} \qquad 3.3(6)$$

As is the case with the centripetal acceleration, the centripetal force is always directed towards the centre of the circular motion. If the centripetal force should cease to exist, the body would leave the circular motion and proceed along a straight line at its existing velocity in accordance with Newton's first law. In other words, the body would abandon the circular orbit tangentially. This phenomenon is illustrated very well by the use of an emery wheel to grind steel. Some of the glowing particles of metal and emery adhere to the wheel and move in circular orbits. Those for which the adhesive force is not sufficient to sustain the circular motion, leave their orbits tangentially.

3.3.3 Examples of centripetal forces

1. Each molecule in a rotating wheel follows a circular path as a result of the intermolecular forces which bind the material of the wheel. If the wheel rotates at an angular velocity for which the forces are insufficient to keep the particles on their circular orbits, the wheel will break.

2. If a motor-car is to make a turn on a horizontal surface, the steering system is used to turn the front wheels to such a position that the frictional force supplies the necessary centripetal force. If the magnitude of the frictional force is insufficient for the car to execute the turn at a given speed, it will move on a circular path of which the radius is larger than that planned by the driver. If the surface is slippery, the coefficient of friction might be too low to allow a turn at a speed which is too great or a radius of curvature which is too small.

3.

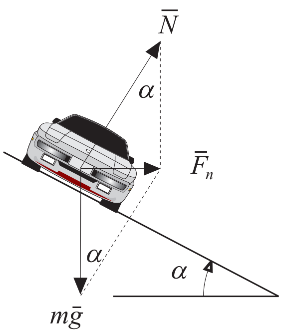


Figure 3.3-2

A well-designed highway allows a vehicle to execute a turn at the correct speed without the use of steering. This is accomplished by sloping the driving surface perpendicular to the driving direction, towards the centre of curvature of the turn. The weight of the vehicle and the normal force of the driving surface on it, supply the necessary centripetal force.

Figure 3.3-2 shows a surface of which the angle of inclination is α. It remains horizontal in the direction of motion. Consider a vehicle which drives at a speed of v m s^{-1} at a position where the radius of curvature is R metre.

If the mass of the vehicle is m kg, its weight is mg newton. Let the magnitude of the normal force which the surface exerts on the car be N newton. From Figure 3.3-2 it may be seen that

$$N \cos \alpha = mg \qquad \text{so that} \qquad N = mg/\cos \alpha$$

The centripetal force is given by

$$F_n = N \sin\alpha = (mg/cos\alpha) sin\alpha = mg \tan\alpha$$

If the vehicle is to execute the turn at v m s^{-1}, then

$$\begin{aligned} F_n &= mg \tan\alpha = mv^2/R \\ \text{so that} \quad \alpha &= \arctan(v^2/Rg) \end{aligned}$$

The designer of a roadway can use this relationship to calculate the correct slope of the surface at a bend for the expected speed. It is of interest to note that the mass of the vehicle does not figure in the calculation. If the driver of a vehicle wishes to execute the turn at a higher speed than that for which the road was designed, he will have to steer towards the inside so that the frictional force between the road and the wheel can contribute towards supplying the necessary centripetal force. If the speed of the vehicle is lower than that for which the road was designed, the driver will have to steer towards the outside of the bend to counteract the excessive centripetal force caused by the slope. Should the driver fail to do this, the vehicle might leave the road towards the inside of the bend.

4. A satellite in a circular orbit around the earth, is kept in its orbit by gravity. Consider a satellite of mass m kg which moves at speed v m s^{-1} in a circular orbit around the earth. The earth's mass is M kg. For a circular orbit the following is valid:

$$mv^2/r = GMm/r^2 \qquad \text{so that} \qquad v^2 = GM(1/r)$$

Each speed corresponds to one orbit only and the mass of the satellite plays no role. When a satellite is to be launched, a choice may be exercised about either the speed of the satellite or the radius of its orbit but not both.

5. According to the Bohr model (Niels Bohr 1913) of the hydrogen atom a single electron revolves around the nucleus in certain allowed circular orbits. In this case the electric force between the nucleus and the electron supplies the centripetal force that maintains the circular motion. The nucleus of a hydrogen atom is a proton with charge $+e$ coulomb and the charge of the electron is $-e$ coulomb. At distance r metre the attraction force between them is ke^2/r^2 newton, in which k is the constant in Coulomb's law. If the mass of an electron is m kg, it follows that

$$mv^2/r = ke^2/r^2 \qquad \text{so that} \qquad v^2 = (ke^2/m)(1/r)$$

in which the relationship between the speed and the orbit radius is similar to that of an earth satellite.

Examples:

1. $g = 10\ \mathrm{m\,s^{-2}}$. A mass is fixed to the end of a light cord of length 0,75 m and swung in a vertical circular orbit. Calculate the minimum speed that the object should have so that the cord just remains taut in the upper position.

In the upper position both the weight and the tension in the cord act downwards, i.e. in the direction of the centre of the orbit. If the cord just remains taut, its tension tends towards zero and the weight is the only force remaining to keep the mass in its orbit. This means that the weight is the centripetal force.

$$mg = mv^2/r \qquad \text{so that} \qquad v = (r \times g)^{\frac{1}{2}} = (7,5)^{\frac{1}{2}} = 2,74\ \mathrm{m\,s^{-1}}$$

2. The mass in problem 1 is 2 kg. Calculate the tension in the cord at the upper position if its speed is $3\ \mathrm{m\,s^{-1}}$.

In this case the tension in the cord is not zero and contributes to the effect of the weight to keep the mass in its orbit, i.e. the sum of the tension and the weight is the centripetal force.

$$T + mg = mv^2/r \qquad \text{so that} \qquad T = (2 \times 3^2/0,75) - 2 \times 10 = 4\ \mathrm{N}$$

3.

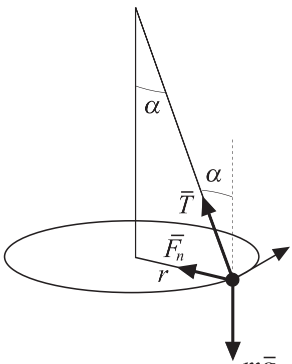


Figure 3.3-3

A **conical pendulum** consists of a small massive ball suspended at the end of a light cord of which the upper point is fixed. The ball moves at $2\ \mathrm{m\,s^{-1}}$ in a horizontal circular orbit of which the radius is 0,5 m. Calculate (a) the angle of the apex of the cone that the cord describes, (b) the length of the cord.

(a) If $\bar{T}$ is the tension in the cord, then

$$\begin{aligned} T &= mg/\cos\alpha \\ \text{and} \quad mv^2/r &= T\sin\alpha = mg\tan\alpha \\ \text{from which} \quad \tan\alpha &= v^2/rg = 4/(0,5 \times 10) \end{aligned}$$

and $\alpha = \arctan 0,8 = 38,66°$ and the angle of the apex is $2\alpha = 77,32°$.

(b) The length of the cord is given by $L = 0,5/(\sin 38,66°) = 0,8$ m.

4. Figure 3.3-4 shows a sketch of the track of a toy motor-car. It has a vertical circular loop of which the radius is 0,1 m.

(a) Calculate the height, h, from which the car should be released so that it will *just* succeed in completing the loop without leaving the track. Friction

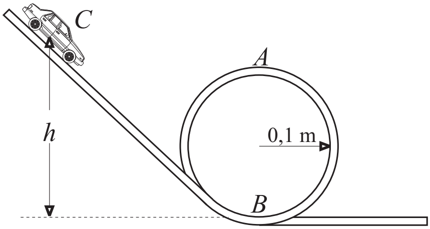


Figure 3.3-4

and the rotational kinetic energy of the wheels may be disregarded. What is the speed of the car at the highest position?

(b) The mass of the car is 50 g. Calculate its speed and the normal force which the track exerts on it in the lowest position if it is released as described in (a).

(a) Since gravity is a conservative force, the mechanical energy of the system is conserved throughout the motion.

$$\begin{aligned} (\text{gravitational} E_p + E_k)_C &= (\text{gravitational} E_p + E_k)_A \\ mgh + 0 &= mg(2r) + \frac{1}{2}mv^2 \\ \text{so that} \quad v^2 &= 2gh - 4gr \qquad \ldots\ldots(1) \end{aligned}$$

in which r represents the radius of the loop. At position A the weight of the car and the normal forces exerted on it by the track, are both downwards and the resultant of these forces is the required centripetal force at this position.

$$N + mg = mv^2/r$$

For the case in question the car must just succeed in staying on the track which means that the normal force, N, which the track exerts on it, tends to zero.

$$\begin{aligned} mg &= mv^2/r \\ \text{or} \quad v^2 &= gr \qquad \ldots\ldots(2) \\ \text{so that} \quad v &= (10 \times 0,1)^{\frac{1}{2}} = 1 \ \mathrm{m\,s^{-1}} \end{aligned}$$

If the above numerical value for v is substituted in Equation (1), the value of h is obtained.

$$h = 2,5r = 0,25 \ \mathrm{m}$$

If the circular track should end at position A, the toy car would proceed on a parabolic curve for free fall. The equation of this curve may be calculated in the same way as was done during the study of projectile motion. It is simple to show that this parabola lies outside the circular track. This means that the car will always complete the circular track if it has enough energy to reach point A.

(b) From conservation of mechanical energy it also follows that

$$\begin{aligned} \frac{1}{2}mv_B^2 &= mgh \\ \text{so that} \quad v_B &= (2gh)^{\frac{1}{2}} = (2 \times 10 \times 0,25)^{\frac{1}{2}} = 2,236 \ \mathrm{m\,s^{-1}} \end{aligned}$$

At position B the normal force of the track on the car is directed upwards and the weight downwards. The normal force, N, minus the weight thus supplies the necessary centripetal force.

$$\begin{aligned} N - mg &= m(v_{_B}^2/r) \\ N &= m(v_{_B}^2/r) + mg \\ &= 0,05(5/0,1) + 0,05 \times 10 \\ &= 3 \text{ N} \end{aligned}$$

3.4 Rotational dynamics

3.4.1 The vectorial properties of angular velocity and angular acceleration

If the angular velocity of a rotating body is known in full, the following three facts about it will be known: (i) The magnitude of the angular velocity. (ii) The orientation of the axis. (iii) The sense of rotation about the axis, i.e. clockwise or counter-clockwise. These three can be specified by means of a single vector, the **angular velocity vector**, $\bar{\omega}$. The representation is unambiguous if the magnitude of the vector represents the magnitude of the angular velocity and its direction represents the direction of the axis according to the corkscrew convention. According to this convention a body will rotate clockwise when viewed in the direction $\hat{\omega} = \bar{\omega}/|\bar{\omega}| = \bar{\omega}/\omega$.

Examples

1. The angular velocity vector of a rotating body is $\bar{\omega} = 12\hat{x}$ rad s^{-1}. The magnitude of the angular velocity is given by $|\bar{\omega}| = \omega = 12$ rad s^{-1} and $\hat{\omega} = \hat{x}$. The body is thus rotating at 12 rad s^{-1} about an axis parallel to $\hat{x}$ in such a way that it rotates clockwise for an observer looking in the direction $\hat{x}$.

2. The angular velocity vector of a rotating body is $\bar{\omega} = -3\hat{z}$ rad s^{-1}. The magnitude of the angular velocity is given by $|\bar{\omega}| = \omega = 3$ rad s^{-1} and $\hat{\omega} = -\hat{z}$. The body is thus rotating at 3 rad s^{-1} about an axis parallel to $-\hat{z}$ in such a way that it rotates clockwise for an observer looking in the direction -$\hat{z}$.

3. The angular velocity of a rotating body is $\bar{\omega} = 2\hat{x} + 3\hat{y} + 6\hat{z}$ rad s^{-1}. Then

$$\begin{aligned} \omega &= (2^2 + 3^2 + 6^2)^{\frac{1}{2}} = 7 \text{ rad s}^{-1} \\ \hat{\omega} &= \bar{\omega}/\omega = \frac{2}{7}\hat{x} + \frac{3}{7}\hat{y} + \frac{6}{7}\hat{z} \end{aligned}$$

The body rotates at 7 rad s^{-1} about an axis parallel to the unit vector $\hat{\omega}$ in such a way that the rotation is clockwise for an observer looking in that direction.

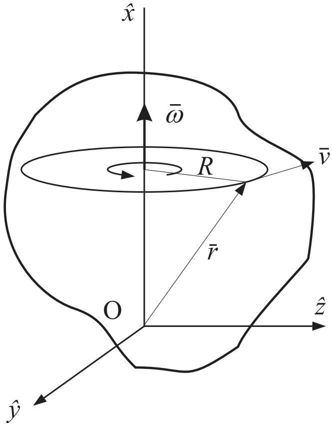


Figure 3.4-1

Figure 3.4-1 shows a body which is rotating about the x-axis at angular velocity $\bar{\omega} = \omega\hat{x}$ rad s^{-1}. Consider a point in the body of which the position vector is $\bar{r}$. The axis of the motion is perpendicular to the circular orbit of this point of which the radius is

$$R = r \sin\phi$$

in which $\phi = \arccos(\hat{\omega}\bullet\hat{r})$ = the angle between the position vector and the angular velocity vector. Note that the velocity vector, $\bar{v}$, of this point is always perpendicular to the plane which contains $\bar{r}$ and $\bar{\omega}$.

The content of Equation 3.2(6) may be stated in words as follows: *The speed of a rotating point is equal to the radius of its circular orbit times the magnitude of the angular velocity.*

Applied to the case under consideration, it means that

$$v = \omega R = \omega r \sin\phi$$

which shows that

$$\bar{v} = \bar{\omega} \times \bar{r} \qquad 3.4(1)$$

In exactly the same way as the angular velocity is represented by a vector, the angular acceleration may also be represented by a vector $\bar{\alpha}$. The axis and sense are determined by the same convention.

The angular acceleration vector is related to the *tangential* linear acceleration vector, $\bar{a}$, by

$$\bar{a} = \bar{\alpha} \times \bar{r} \qquad 3.4(2)$$

The directions of $\bar{\omega}$ and $\bar{\alpha}$ are determined by a convention. This has the result that they have different **transformation properties** than conventional vectors such as position, velocity, acceleration etc. which have been commonly used up to now and which are known as **true vectors**. The signs of the components of true vectors change differently to those of $\bar{\omega}$ and $\bar{\alpha}$ under the transformation known as the **inversion of axes**. For this reason the vectors representing angular velocity and angular acceleration are called **pseudovectors** or **axial vectors**. In the study of vector algebra it should become clear that the vector product of any two real vectors is a pseudovector. The vector product of a pseudovector and a true vector is a true vector.

3.4.2 The torque of a force

The sketches in Figure 3.4-2 show a wheel which is constrained to turn about a fixed axis, O. One and the same force, $\bar{F}$, is applied in different ways along the perimeter in efforts to cause rotation. In each case the line along which the force acts, lies in the plane containing the wheel.

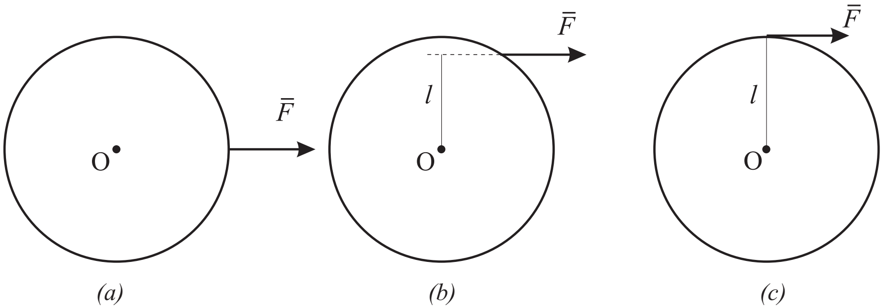


Figure 3.4-2

It is common knowledge that the force as applied in sketch (a), will not cause the wheel to rotate. It has no **rotatory effect** about O. The force applied as shown in (b) will be able to cause rotation about O, but it will be less than that illustrated in (c). Over and above the magnitude and direction of a force, its rotatory effect also depends upon the perpendicular distance from its line of action to the axis. This distance is called the **lever arm** or **moment arm**.

The ability to cause a change in the rotational condition of a body by a given force, is measured by the **torque** or **moment** of the force about the given axis. The torque or moment, τ, of a force about a given point is defined as its magnitude times the lever arm. The length of the lever arm is indicated in Figure 3.4-2 by symbol l.

$$\tau = \text{magnitude of the force} \times \text{lever arm} = F \times l$$

Besides the magnitude of a torque, its *sense* (clockwise or counter-clockwise) is also of importance. In sketch (c) the force will tend to cause clockwise rotation. If the force were applied at the lower edge of the wheel whilst keeping its direction, it would tend to cause rotation in a counter-clockwise sense.

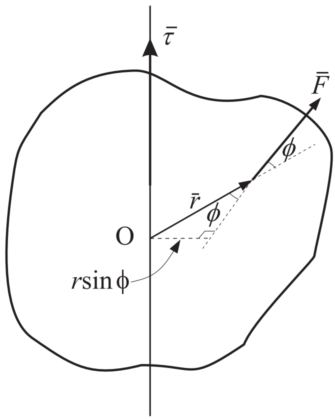


Figure 3.4-3

As is the case with angular velocity and angular acceleration, it is possible to represent torque by a vector. Consider the force $\bar{F}$ shown in Figure 3.4-3 which acts on a body of which the motion is limited to rotation about a fixed axis, O. *The line of action of the force lies in the plane which contains the circle which its point of application describes.* Choose a frame of reference with origin at the centre of the circle which the point of application of the force describes. The position vector of the point of application of the force is $\bar{r}$ and the angle between $\bar{r}$ and $\bar{F}$ is ϕ. The length of the lever arm is $l = r\sin\phi$ and the magnitude of the torque is given by

$$\tau = Fr\sin\phi \qquad 3.4(3)$$

A **torque vector**, τ, is chosen in the same direction as the angular acceleration that it will cause when applied. As a result of this choice, it should be clear that Equation 3.4(3) may be rewritten as a vector equation.

$$\bar{\tau} = \bar{r} \times \bar{F} \qquad 3.4(4)$$

This equation will be used in future as the definition of torque. As angular velocity and angular acceleration, torque is also a pseudovector. Its units are newton-meter (N m) and the dimensions are given by $[\tau] = [\mathrm{M\,L^2T^{-2}}]$.

In the preceding explanation it was stressed that the force acts in a plane which is perpendicular to the axis. It was chosen in this way so that the torque vector could be parallel to the axis. This, of course, is a very special case and the question arises as to what would occur if the force vector were not in such a plane. In an extreme case where the force is parallel to the axis, it would have no effect on the rotational condition of the body. This will be the result of the constraints caused by the fixed axis. The force will still have a torque as given by Equation 3.4(4), but it will be counterbalanced by the torque of the constraining force in the fixed axis.

The torque of a force with any direction will be given by Equation 3.4(4), but only its component parallel to the fixed axis will be able to change the rotational condition of the body. The torque of the constraining force in the axis will balance the component of the torque perpendicular to the axis.

Bodies which are constrained to rotate about a fixed axis will be the main scope of this study and for this reason mainly torque vectors which are parallel to the axis will be considered. This reduces problems on rotational dynamics to one dimension.

In Figure 3.4-2 only one force acting upon the wheel in each case and its effect on the rotation were considered . The question arises why the force did not cause the wheel to translate. The answer is found in the constraining force at the axle. Since the wheel was in translation equilibrium in each case (its centre of mass remained at rest) the resultant of the applied force and the constraining force is zero. This shows that the force at the axle is equal in magnitude but opposite in direction to the applied force.

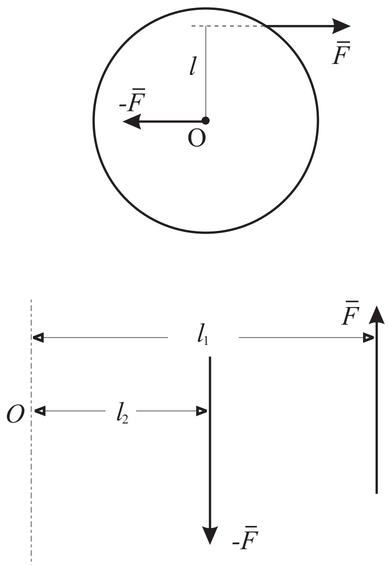


Figure 3.4-4

In general the lines along which these two forces act, do not coincide and they form a so-called **couple** as shown in Figure 3.4-4(a). A couple always consists of two forces of equal magnitude which are opposite in direction but do not act along the same straight line. The magnitude of the torque of a couple is defined as the product of the magnitude of one of the two forces and the perpendicular distance between their lines of action.

$$\tau = F \times l \qquad 3.4(5)$$

It is simple to illustrate that the torque of a couple is independent of the position around which it is calculated. Figure 3.4-4(b) illustrates the calculation of the magnitude of the torque about point O.

$$\tau = F \times l_1 - F \times l_2 = F(l_1 - l_2)$$

This result is independent of the position of point O. In this calculation the signs of the two moments are opposite because their rotational effects about O are in opposite senses (clockwise and counter-clockwise).

The following theorem is useful in the solution of problems in mechanics: *The effects of all forces acting on a body can be reduced to one single force plus a couple.* The proof of this is left to the reader as an exercise.

In chapter 2 the spatial properties of the bodies were irrelevant because they were considered to be point masses. It now becomes apparent that even though the resultant of a number of forces acting on a body might be zero, a torque might still exist. This requires a revision of the equilibrium conditions of a body. It may be formulated as follows: *When a body is in static equilibrium, the resultant of all the forces acting on it, is equal to zero. The same applies to the torques about any point, of all these forces.* The use of these equilibrium conditions is illustrated in the following examples.

Examples

1.

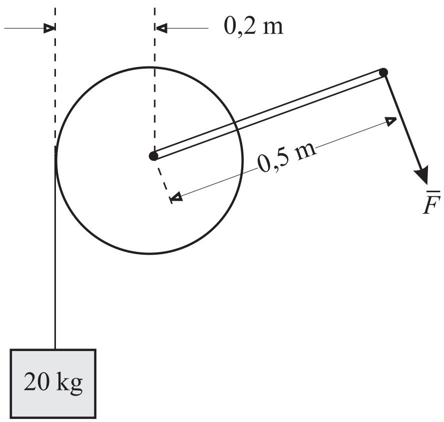


Figure 3.4-5

The radius of a windlass is 0,2 m and the length of its crank 0,5 m. It is used to hoist a mass of 20 kg which is suspended from a light rope of which the mass may be disregarded. $g = 10$ m s^{-2}. Calculate the magnitude of the force that should be applied at the handle at right angles to the crank, to keep the load in equilibrium.

The magnitude of the counter-clockwise torque is

$$\begin{aligned}\text{force} \times \text{lever arm} &= (20 \times 10) \times 0,2 \\ &= 40 \text{ N m}\end{aligned}$$

If F is the magnitude of the applied force, the magnitude of the clockwise torque is

$$\text{force} \times \text{lever arm} = F \times 0,5 = 0,5F$$

For equilibrium, these two torques must have the same magnitude. From this follows that

$$F = 80 \text{ N}$$

2. Two men, A and B, use a uniform homogeneous pole with length 4 m and mass 10 kg to carry a load of 50 kg. The ends of the pole rest on their shoulders and the load hangs 2,5 m from A's shoulder. The centre of mass of the pole is at its geometric centre. Calculate the forces F_A and F_B that the two men have to exert respectively to keep the pole with the load on it horizontal. $g = 10$ m s^{-2}.

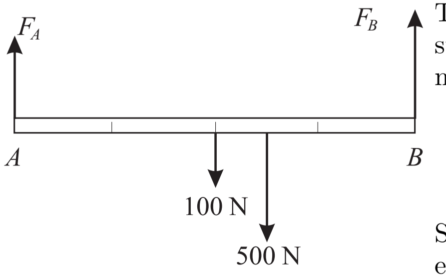


Figure 3.4-6

To keep the system in equilibrium, the sum of all the forces acting on the pole must be equal to zero.

$$\begin{aligned} F_A + F_B &= 10 \times 10 + 50 \times 10 \\ &= 600 \qquad \ldots\ldots(1) \end{aligned}$$

Since the system is also in rotational equilibrium, the sum of all the torques *about any point* is equal to zero. Calculate the torques about A.

$$4F_B = 2 \times 100 + 2,5 \times 500$$

so that

$$F_B = 362,5 \text{ N}$$

If the numerical value of F_B is substituted in Equation (1), the value of F_A may be calculated.

$$F_A = 237,5 \text{ N}$$

It is suggested that the reader solve the problem by calculating the torques about a point other than A. The chosen point need not be on the pole. The result will be the same. It does, however, always simplify the equations if the torques are calculated about a point through which one of the unknown forces acts.

3.4.3 The conservation of angular momentum

The **angular momentum**, $\bar{J}$ of a point mass with respect to the origin of an inertial frame of reference, is defined as

$$\bar{J} = \bar{r} \times \bar{p} = \bar{r} \times m\bar{v} \qquad 3.4(6)$$

in which $\bar{r}$ is the position vector of the point mass and $\bar{p}$ its linear momentum in the given frame of reference. Angular momentum is also known as the **moment of the momentum**. Its units are $\text{kg m}^2\text{s}^{-1}$ which are the same as joule-second (J s). $[J] = [\text{M L}^2\text{T}^{-1}]$.

In accordance with the definition of a vector product, the direction of angular momentum is perpendicular to the plane containing the position vector and the velocity vector of the point mass. The three vectors, $\bar{J}$, $\bar{r}$ and $\bar{v}$ form a right-handed system. The component of $\bar{J}$ along any line (axis) through the position which was chosen as the origin, is often called the angular momentum of the point mass *about this axis.* Note that angular momentum is defined relative to a given *point.*

If a force $\bar{F}$ acts on the point mass, its velocity and position and therefore its angular momentum will change. To investigate this change, the time derivative of the angular momentum is calculated.

$$\begin{aligned} & \frac{\bar{J}}{dt} &=& \frac{d}{dt}(\bar{r}\times\bar{p}) = \frac{\bar{r}}{dt}\times\bar{p} + \bar{r}\times\frac{d\bar{p}}{dt} \\ \text{but} \quad & \frac{d\bar{r}}{dt}\times\bar{p} &=& \bar{v}\times m\bar{v} = \bar{0} \\ \text{and} \quad & \bar{r}\times\frac{d\bar{p}}{dt} &=& \bar{r}\times\bar{F} = \bar{\tau} \qquad \text{Newton 2: } \bar{F} = d\bar{p}/dt \end{aligned}$$

This leads to the following important result:

$$\tau = \frac{d\bar{J}}{dt} \tag{3.4(7)}$$

This result may be formulated as follows: *The torque acting on a point mass is equal to the rate at which its angular momentum changes.* The point about which the torque is calculated, is the same as that relative to which the angular momentum is measured. This result is the rotation analogue of Newton's second law for translational motion, viz. $\bar{F} = d\bar{p}/dt$.

If the torque in Equation 3.4(7) is zero, then $d\bar{J}/dt = \bar{0}$ and $\bar{J}$ will be a constant vector. This is a special case of the principle of **conservation of angular momentum**. The angular momentum of a point mass remains constant if no unbalanced external torque acts on it.

A **central force** is one which is either parallel or anti-parallel to the position vector of the particle upon which it acts. The source of the force is situated at the origin. Gravitation between two point masses, electrostatic forces between point charges and nuclear forces between nucleons, are examples of central forces. All central forces have the following mathematical form:

$$\bar{F} = F(r)\hat{r} \tag{3.4(8)}$$

in which the scalar function $F(r)$ may be positive or negative, depending on whether the force is an attractive force or a repulsion. The torque of a central force about the origin (as defined previously) is always equal to zero,

$$\bar{\tau} = \bar{r}\times\bar{F} = \bar{r}\times F(r)\hat{r} = \bar{0}$$

which leads to the important result that the angular momentum of a point mass cannot change under the action of a central force. Since angular momentum is a vector, it also implies that the direction will remain constant and that the motion occurs in a plane. Although not formulated in this way, this result was discovered by Kepler during his studies concerning the motion of planets. This helped Newton to develop his laws of motion and also his law of gravitation.

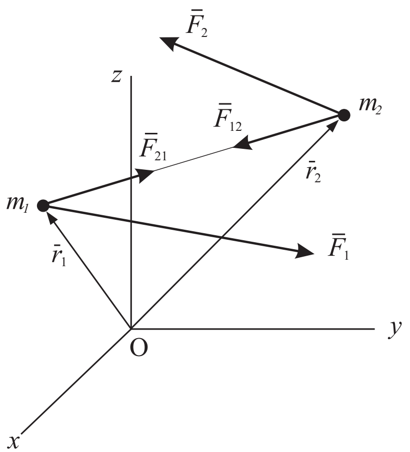


Figure 3.4-7

It is often necessary to deal with the angular momentum of rigid bodies rather than that of point masses. In order to obtain the necessary information, the simplest case of a body, i.e. one consisting of two point masses only, will be studied.

Consider two point masses of which the position vectors are $\bar{r}_1$ and $\bar{r}_2$ and the external forces which act on them are $\bar{F}_1$ and $\bar{F}_2$ respectively. The point masses exert forces $\bar{F}_{12}$ and $\bar{F}_{21}$ on each other.

For each of the point masses Equation 3.4(7) applies.

$$\tau_1 = \frac{d\bar{J}_1}{dt} \qquad \text{and} \qquad \tau_2 = \frac{d\bar{J}_2}{dt}$$

The sum of these two equation gives

$$\tau_1 + \tau_2 = \frac{d}{dt}(\bar{J}_1 + \bar{J}_2)$$

The torques, τ_1 and τ_2, which act on the two point masses may now be calculated as follows:

$$\begin{aligned} \bar{\tau}_1 &= \bar{r}_1 \times (\bar{F}_1 + \bar{F}_{21}) = \bar{r}_1 \times \bar{F}_1 + \bar{r}_1 \times \bar{F}_{21} \\ \bar{\tau}_2 &= \bar{r}_2 \times (\bar{F}_2 + \bar{F}_{12}) = \bar{r}_2 \times \bar{F}_2 + \bar{r}_2 \times \bar{F}_{12} \end{aligned}$$

The vector $\bar{r}_1 - \bar{r}_2$ is parallel to the line which joins the two particles. The two forces described by Newton's third law, are always parallel to this line and therefore the vector has to be zero. The total torque is thus

$$\begin{aligned} \bar{\tau}_1 + \bar{\tau}_2 &= \bar{r}_1 \times \bar{F}_1 + \bar{r}_2 \times \bar{F}_2 = \bar{\tau}_e \\ &= \text{total external torque on the system} \end{aligned}$$

This argument may be extended to any number of particles in the body under consideration. This gives the result

$$\bar{\tau}_e = \frac{d\bar{J}}{dt} \qquad 3.4(9)$$

in which $\bar{\tau}_e$ is the total external torque on the system of point masses and $\bar{J}_1 + \bar{J}_2 + \bar{J}_3 + \ldots = \sum_i \bar{J}_i = \bar{J} =$ its total angular momentum about the origin of the frame of reference.

This result emphasises two important facts: (i) The torques of internal forces cannot have an influence on the angular momentum of a system of point masses. (ii) The time-rate of change in the angular momentum of such a system about a given point, is equal to the total torque of the external forces about the same point. Equation 3.4(9) is the adaptation of Newton's second law of motion for rotational motion of a system of point masses.

If the total external torque is equal to zero, then $d\bar{J}/dt = \bar{0}$. Integration of this equation leads to the principle of conservation of angular momentum for a system of point masses.

$$\bar{J} = \sum_i \bar{J}_i = \text{ a constant vector if } \bar{\tau}_e = \bar{0} \qquad 3.4(10)$$

Since the angular momentum of a system can be changed by an *external* torque only, the subscript e will be omitted in the rest of this book. Whenever $\bar{\tau}$ appears, it may be assumed to be an *external* torque.

3.4.4 The angular momentum of a rigid body about a given axis

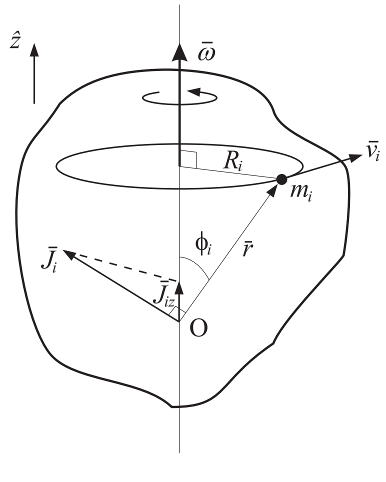


Figure 3.4-8

Consider a rigid body which rotates about the z-axis of a Cartesian frame of reference with an angular velocity of $\bar{\omega} = \omega\hat{z}$. What applied to a collection of point masses in 3.4.3 may also be applied to a rigid body. Within a rigid body rotating about a fixed axis, the internal forces and the constraining forces in the axle cause the constituent particles to move in circular orbits of which the centres are on the axis. This simplifies the problem to a large extent. The radius of the circular orbit of particle number i in Figure 3.4-8, is given by

$$R_i = r_i \times \sin\phi_i$$

in which ϕ is the angle between its position vector, $\bar{r}_i$, and the axis, $\hat{z}$. The velocity vector of the particle is given by

$$\bar{v}_i = \bar{\omega} \times \bar{r}_i$$

in which the angular velocity, $\bar{\omega}$ is the same for all particles since the body is rigid. The speed of the particle is the magnitude of its velocity vector and is given by

$$v_i = \omega r_i \sin\phi_i = \omega R_i$$

Relative to the origin, the angular momentum of the particle is given by

$$\bar{J}_i = m_i \bar{r}_i \times \bar{v}_i$$

This vector is perpendicular to the plane containing the position vector, $\bar{r}_i$, and the velocity vector, $\bar{v}_i$, and is thus at an angle of $\pi/2 - \phi_i$ with the z-axis. The magnitude of its angular momentum is thus

$$J_i = m_i r_i \sin\pi/2 = m_i r_i v_i$$

and its component parallel to the z-axis,

$$\begin{aligned} J_{iz} &= m_i r_i v_i \cos(\pi/2 - \phi_i) = m_i r_i v_i \sin\phi_i \\ &= m_i (r_i \sin\phi_i)(\omega R_i) = m_i R_i^2 \omega \end{aligned}$$

For the whole body the component of its total angular momentum in the direction of the z-axis, is

$$\begin{aligned} J_z &= J_{1z} + J_{2z} + J_{3z} + \ldots = \sum_i J_{iz} \\ &= (m_1 R_1^2 + m_2 R_2^2 + m_3 R_3^2 + \ldots)\omega \\ &= (\sum_i m_i R_i^2)\omega \end{aligned} \qquad 3.4(11)$$

in which the important quantity

$$I = \sum_i m_i R_i^2 \qquad 3.4(12)$$

is known as the **moment of inertia** of the body about the given axis. Equation 3.4(11) may now be written as

$$J_z = I\omega \qquad 3.4(13)$$

The total angular momentum vector of the body about the given axis, is given by

$$\bar{J} = \sum_i \bar{J}_i$$

and is in general not parallel to the rotation axis. It can be shown that each rigid body has at least three mutually perpendicular axes for which the angular momentum vector (and thus the angular velocity vector) will be parallel to the axis of rotation. Such axes are known as **principal axes** and the moments of inertia about them, the **principal moments of inertia**. It can further be shown that if a body is homogeneous and symmetrical about an axis, that axis will be a principal axis. If a body thus rotates about a principal axis, scalar equation 3.4(11) may be written as a vector equation.

$$\bar{J} = I\bar{\omega} \qquad 3.4(14)$$

Equation 3.4(14) applies to rotation about a principal axis only.

When a wheel of a motor-car is to be balanced, the intention is to distribute its mass in such a way that the existing fixed axis will be a principal axis. If a wheel is balanced *statically*, it will be coincidental if this is the case. For this reason motor-car wheels are always balanced *dynamically*. If a wheel is dynamically out of balance, the angular momentum vector will **precess** about the axis even if it is in static balance. A precessing angular momentum vector always makes an angle with the axis and rotates on a conical surface about the axis during one revolution. Unbalanced motor-car wheels cause not only irregular wear of the tyres but also a dangerous jerking of the steering which might lead to loss of control over the vehicle.

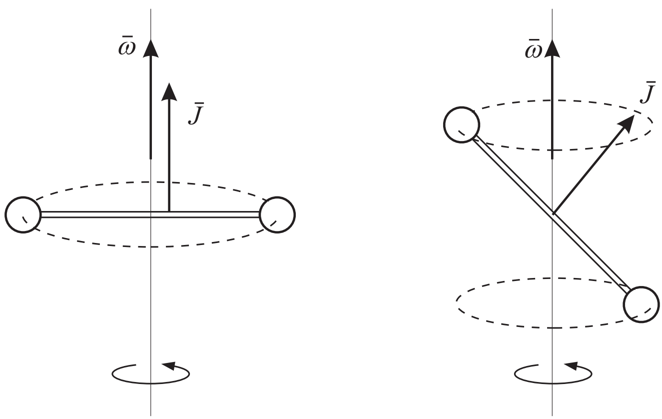


Figure 3.4-9

The difference between static and dynamic balance is illustrated by the dumb-bell shaped body in Figure 3.4-9. In Figure 3.4-9(a) the body rotates about a principal axis and the angular momentum vector is parallel to the angular velocity vector. In Figure 3.4-9(b) the body does not rotate about a principal axis and these two vectors are not parallel. From the sketches it should be clear that static balance exists in both cases but that dynamic balance exists for case (a) only. (If the axis is horizontal, the body will be in equilibrium in any position for both cases. This implies that the centre of mass is on the axis.)

In section 3.5, the moments of inertia of a number of regular bodies will be calculated. Before that is done, attention will be given to the equation for the rotation of a body and its rotational energy.

3.4.5 The equation for the rotation of a rigid body

Equation 3.4(9) gives the equation of motion for a collection of point masses.

$$\bar{\tau} = \frac{d\bar{J}}{dt}$$

Since internal forces cannot influence the angular momentum, this equation also applies to a rigid body. For the special case where the body rotates about a *principal axis*, the following applies:

$$\tau = \frac{d\bar{J}}{dt} = \frac{d}{dt}(I\bar{\omega}) = I\frac{d\bar{\omega}}{dt} = I\bar{\alpha} \qquad 3.4(15)$$

This equation is Newton's second law of motion adapted for the rotational motion of a body about a principal axis. In this equation, the torque vector is the analogue of force in translational motion, moment of inertia the analogue of mass and angular acceleration that of translational acceleration.

When the external torque is equal to zero, the total angular momentum which is given by $I\bar{\omega}$ for rotation about a principal axis, is a constant vector. For a rigid body, the moment of inertia about a fixed axis cannot change and the angular velocity will remain constant in the absence of an external torque.

If a body is not rigid, the moment of inertia about the axis of rotation is not constant. If such a body rotates with no external torque acting on it and the moment of inertia is decreased, the angular velocity will increase and vice versa. Examples: (i) The angular velocity of a diver performing a somersault increases when his legs are drawn in towards his chest with the resulting decrease in his moment of inertia. (ii) A figure skater performing a pirouette on one skate, can manipulate his angular velocity by the extension and contraction of the free leg and his arms. (iii) By manipulating his moment of inertia, a trampolinist or a falling cat can turn around in mid-air in the absence of an external torque.[5]

When a body is not rotating about a principal axis, the following will be valid for the axis of rotation (e.g. the z-axis):

$$\frac{dJ_z}{dt} = I\frac{d\omega}{dt} = I\alpha = \tau_z \qquad 3.4(16)$$

Comment: Problems regarding rigid bodies rotating about fixed axes, especially principal axes, are usually very simple. Since all the vector quantities concerned are parallel, the use of vector notation is superfluous. Such rotational problems are analogous to translational problems in one dimension.

Examples:

1.

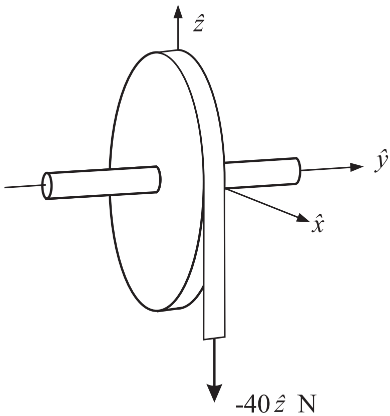


Figure 3.4-10

The solid cylindrical wheel in Figure 3.4-10 has a radius of 0,5 m. It can rotate about a light axis of which the friction may be disregarded. The moment of inertia of the wheel about this axis is 50 kg m^2. By means of a thin unstretchable ribbon which is wound around the wheel, a force of $-40\hat{z}$ N is applied tangentially to the perimeter. (a) Calculate the torque, $\bar{\tau}$, of the force about the axis. (b) Calculate the angular acceleration, $\bar{\alpha}$, of the wheel. (c) The initial angular velocity of the wheel is $\bar{\omega}(0) = -10\hat{y}$ rad s^{-1}. Calculate its angular velocity, $\bar{\omega}(t)$, as a function of time. (d) Calculate $\theta(t)$, its rotation angle as a function of time. Assume that $\theta(0) = 0$.

(a) The position vector of the point of application of the tangential force is $\bar{r} = 0,5\hat{x}$ m. Therefore

$$\bar{\tau} = \bar{r} \times \bar{F} = (0,5\hat{x}) \times (-40\hat{z}) = 20\hat{y} \ \text{N m}$$

(b) The torque which was calculated in (a), causes an angular acceleration of the wheel which is given by

$$\bar{\alpha} = \bar{\tau}/I = (20\hat{y})/(50) = 0,4\hat{y} \ \text{rad s}^{-2}$$

[5]See *ARCHIMEDES*, October 1971. Published by the Foundation for Education, Science and Technology

(c)

$$\begin{aligned}\Delta\bar{\omega} = \int_0^t \bar{\alpha} dt &= \int_0^t (0, 4\hat{y}) dt \\ \text{from which follows} \quad \omega &= (0, 4t - 10)\hat{y} \text{ rad}\end{aligned}$$

(d)

$$\theta = \int_0^t \omega dt = \int_0^t (0, 4t - 10) dt = (0, 2t^2 - 10t) dt \text{ rad}$$

Initially the wheel turns counter-clockwise if observed in the direction $\hat{y}$ whilst the torque acts in the opposite sense. After 25 seconds it comes to rest and then turns clockwise. The fact that $\theta = 0$ at $t = 50$ s, means that it had then turned through equal angles clockwise and counter-clockwise.

2. The wheel in the Figure 3.4-11 is the same as that in the previous problem. A mass of 2 kg is suspended from the end of the ribbon. $g = 10 \text{ m s}^{-2}$. The system is initially at rest. The mass is allowed to accelerate by the action of gravity and thereby setting the wheel in motion. Calculate (a) the tension in the ribbon, (b) the linear acceleration of the mass, (c) the angular acceleration of the wheel, (d) the magnitude of the torque which acts upon the wheel while it is accelerating.

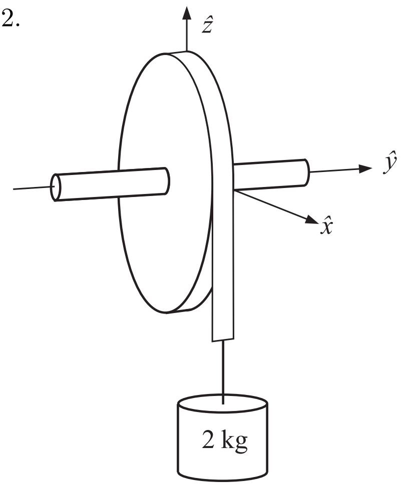


Figure 3.4-11

Let the magnitude of the tension in the ribbon be T newton. The magnitude of the downward force on the 2 kg mass is then $(2 \times 10 - T)$ newton. This force accelerates the mass according to Newton's second law.

$$20 - T = 2a \qquad \text{so that} \qquad T = 20 - 2a \ldots\ldots(1)$$

The tension in the ribbon has a torque about the axis which is given by

$$\tau = 0,5T \qquad \ldots\ldots(2)$$

This torque causes an angular acceleration, α, in the wheel.

$$0,5T = 50\alpha \qquad \text{so that} \qquad T = 100\alpha \qquad \ldots\ldots(3)$$

Since the ribbon is tied firmly to the perimeter of the wheel and does not stretch, we have

$$\alpha = a/0,5 = 2a \qquad \ldots\ldots(4)$$

These four equations may be solved for the four quantities which were to be calculated.
$T = 2000/101 = 19{,}80$ N; $a = 0{,}0990\ \mathrm{m\,s^{-2}}$; $\alpha = 0{,}1980\ \mathrm{rad\,s^{-2}}$; $\tau = 9{,}901$ N m.

3.4.6 The rotational kinetic energy of a rigid body rotating about a fixed axis

Consider a rigid body rotating about the $\hat{z}$-axis of a Cartesian frame of reference at angular velocity $\bar{\omega}$. Particle number i describes a circular orbit with radius R_i as shown in Figure 3.4-8. As derived previously, the speed of the particle is given by

$$v_i = \omega R_i$$

The kinetic energy of the molecule is given by

$$(E_k)_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i(\omega R_i)^2 = \frac{1}{2}(m_i R_i^2)\omega^2 = \frac{1}{2}I_i\omega^2 \qquad 3.4(17)$$

in which I_i is the moment of inertia of the particle about the axis of rotation. This shows that the kinetic energy of a point mass on a circular orbit is given by either $\frac{1}{2}mv^2$ or $\frac{1}{2}I\omega^2$.

The rotational kinetic energy of the entire body is given by the sum of the kinetic energies of the constituent particles.

$$E_k = \sum_i \frac{1}{2}(m_i R_i^2)\omega^2 = \frac{1}{2}I\omega^2 \qquad 3.4(18)$$

in which I is the moment of inertia of the body about the axis of rotation. The units and dimensions are the same as for any other form of energy.

3.4.7 Work and power in terms of torque

In rotational problems it is convenient to express work and power in terms of torque. This discussion will be limited to cases in which work is done on rigid bodies rotating about a fixed axis. For such limitations, which simplify problems to a large extent, it will be necessary to consider only forces (or their components) which act in planes perpendicular to the axis. Let $\bar{F}$ be such a force. Its point of application is restricted to circular motion about the axis. The position vector of the point of application is $\bar{r}$. The force may be resolved into two components: $\bar{F}_s$ tangential to the circle along which the point of application moves and $\bar{F}_n$, the component perpendicular to it.

Since the body on which work is done is rigid, no radial displacement of the point of application occurs and, accordingly, the normal or radial component, $\bar{F}_n$, does no work. The tangential component, however, does work which may be written as

$$dA = \bar{F} \bullet \bar{ds} = F_s ds$$

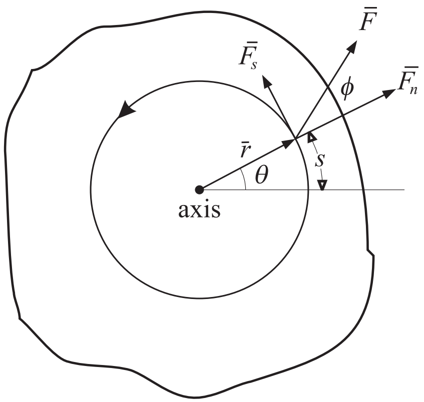


Figure 3.4-12

If the point of application moves a distance ds along the circle, it corresponds with a rotation of $d\theta = ds/r$. The work done, is given by

$$dA = F_s r\, d\theta$$

But $F_s r = (F \sin\phi) r = F(r \sin\phi) =$ the magnitude of the torque of the applied force about the axis. In this, ϕ is the angle between $\bar{F}$ and $\bar{r}$. Equation 3.4(3) which applies to Figure 3.4-3, explains this relationship. From figure 3.4-12 it should be clear that

$$dA = \tau d\theta \qquad 3.4(19)$$

If the body rotates through an angle of θ, the work done is given by

$$A = \int_0^\theta \tau\, d\theta \qquad 3.4(20)$$

For the calculation of power, the work may be written as a function of time and the time-derivative calculated. That will give the power as a function of time. From Equation 3.4(19) a useful relationship may be derived.

$$P = \frac{dA}{dt} = \tau \frac{d\theta}{dt} = \tau\omega \qquad 3.4(21)$$

Suppose a rigid body with moment of inertia I about its constraining axis has an initial angular velocity of ω_1. A torque τ acts on it and after completion of rotation through angle θ, its angular velocity is ω_2. In this case it may written that

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta}$$

in which the chain rule for differentiation was used.. From this follows

$$\tau d\theta = I\omega d\omega$$

Integration of this differential equation gives the following result:

$$A = \int_0^\theta \tau d\theta = \int_{\omega_1}^{\omega_2} I\omega d\omega = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2 \qquad 3.4(22)$$

This result corresponds exactly to that shown in equation 2.5(3) in which work done against translational inertia (mass) is equal to the increase in translational kinetic energy. In this case it is the increase in rotational kinetic energy of the given body about a specified axis and which is defined by Equation 3.4(18).

3.4.8 Combined translation and rotation

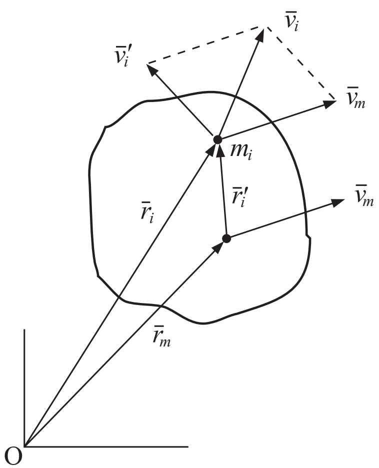


Figure 3.4-13

Figure 3.4-13 shows a rigid body of which the centre of mass is at position $\bar{r}_m$ relative to a given inertial frame of reference. Consider particle number i with mass m_i and position vector $\bar{r}_i'$ *relative to the centre of mass of the body.* If the position of this particle relative to the inertial frame of reference is $\bar{r}_i$, it follows from the Galilei transformation that

$$\bar{r}_i = \bar{r}_m + \bar{r}_i'$$

The time-derivative of this equation gives

$$\bar{v}_i = \bar{v}_m + \bar{v}_i'$$

in which $\bar{v}_m$ is the velocity of the centre of mass relative to the inertial system, $\bar{v}_i$ the velocity of molecule number i relative to the same system and $\bar{v}_i'$ that of the molecule relative to the centre of mass of the body.

The kinetic energy of the particle relative to the inertial frame of reference is given by

$$\begin{aligned}(E_k)_i &= \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i \bar{v}_i \bullet \bar{v}_i \\ &= \frac{1}{2}m_i(\bar{v}_m + \bar{v}_i')\bullet(\bar{v}_m + \bar{v}_i')\end{aligned}$$

$$= \frac{1}{2} m_i (v_m^2 + v_i'^2 + 2\bar{v}_m \bullet \bar{v}_i')$$

The sum of the kinetic energies of all the particles gives the total kinetic energy of the entire body.

$$E_k = \sum_i \frac{1}{2} m_i (v_m^2 + v_i'^2 + 2\bar{v}_m \bullet \bar{v}_i')$$

which may be rewritten as

$$E_k = \frac{1}{2}(\sum_i m_i) v_m^2 + \frac{1}{2}(\sum_i m_i v_i'^2) + \sum_i m_i \bar{v}_m \bullet \bar{v}_i'$$

in which the summation includes all the particles of the body. The last term of this expression may be written as follows:

$$\sum_i m_i \bar{v}_m \bullet \bar{v}_i' = \bar{v}_m \bullet \sum_i m_i \bar{v}_i' = \bar{v}_m \bullet \sum_i m_i \frac{d\bar{r}_i'}{dt}$$

Since the masses, m_i, are time-independent, this term may be written as

$$\bar{v}_m \bullet \sum_i m_i \frac{d\bar{r'}_i}{dt} = \bar{v}_m \bullet \frac{d}{dt}(\sum_i m_i \bar{r}_i')$$

Since the position vectors $\bar{r}_i$ are relative to the centre of mass of the body, the sum in brackets is equal to zero. (See Equation 3.1(1) for the definition of centre of mass. According to the said sum, the position of the centre of mass coincides with the origin and is thus equal to the zero vector.)

The first term in the expansion of the kinetic energy, contains the factor $\sum_i m_i$ in which the summation includes all the particles of the body. This is simply the mass of the body and will henceforth be represented by M. The total kinetic energy of the body is

$$E_k = \frac{1}{2} M v_m^2 + \frac{1}{2} \sum_i m_i v_i'^2$$

The vector $\bar{v}_i'$ is the velocity of particle number i relative to the centre of mass. Since the body is rigid, this is a velocity due to rotation of the body about an axis through the centre of mass. The speed v_i' is given by $R_i\omega$ in which R_i is the distance between the particle and the axis (see Figure 3.4-8) and ω the magnitude of the angular velocity. From this follows that

$$\frac{1}{2} \sum_i m_i v_i'^2 = \frac{1}{2}(\sum_i m_i R_i^2)\omega^2 = \frac{1}{2} I \omega^2$$

in which I is the moment of inertia of the body about the rotation axis through the centre of mass. The total kinetic energy of the body may now be written as

$$E_k = \frac{1}{2}Mv_m^2 + \frac{1}{2}I\omega^2 \qquad 3.4(23)$$

This is a most important result which states that the total kinetic energy of a moving body may be written as the sum of two terms: (i) The kinetic energy of translation of a point mass which has the same mass as the body and which has the same speed as its centre of mass, and (ii) the rotational kinetic energy of the body about an axis through its centre of mass. If the centre of mass executes rotational motion about a point, the first term may be written in the same form as the second. The validity of this statement is shown in Equation 3.4(17).

Examples:

1.

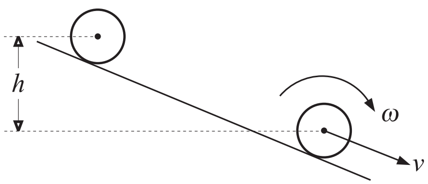


Figure 3.4-14

The moment of inertia of a solid homogeneous sphere about an axis through its centre is $0,4MR^2$ in which M is its mass and R its radius. (The formula will be deduced in section 3.5). A solid homogeneous sphere with mass M kilogram and radius R, rolls without slipping down an inclined plane. Calculate the speed of its centre of mass if it starts from rest and descends a vertical distance h metre while rolling down. The set-up is shown in Figure 3.4-14.

From the principle of energy conservation, the loss of gravitational potential energy will be equal to the gain in kinetic energy. The loss in gravitational potential energy is $\Delta E_p = Mgh$ and the gain in kinetic energy, $\Delta E_k = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ in which v is the speed of the centre of mass and ω the angular velocity about an axis through the centre of mass. Since it rolls without slipping, a simple relationship exists between the speed of the sphere's centre of mass and its angular velocity about the centre of mass: $v = R\omega$.

From the above follows

$$\begin{aligned} \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 &= Mgh \\ \frac{1}{2}Mv^2 + \frac{1}{2}(0,4MR^2)(v/R)^2 &= Mgh \\ \text{so that} \qquad v &= (10gh/7)^{\frac{1}{2}} = 1,195(gh)^{\frac{1}{2}} \end{aligned}$$

2. Consider the same set-up as that in problem 2 in section 3.4.5 which is shown

in Figure 3.4-11. Use the principle of conservation of energy and calculate the speed of the 2 kg mass after descending vertically from rest through 4 m.

The loss in gravitational potential energy of the 2 kg mass is equal to the gain in kinetic energy of the system. The kinetic energy consists of the translational energy of the 2 kg and the rotational energy of the wheel about its axis.

The change in gravitational potential energy of the 2 kg is given by

$$\Delta E_p = 2 \times 10 \times 4 = 80 \text{ J}$$

The gain in kinetic energy is

$$\Delta E_k = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2} \times 2 \times v^2 + \frac{1}{2} \times 50 \times \omega^2$$

If the ribbon does not stretch or slip, $\omega = v/0,5 = 2v$. From this follows

$$v^2 + 25(2v)^2 = 80 \qquad \text{so that} \qquad v = 0,8900 \text{ m s}^{-1}$$

3. The moment of inertia of a solid homogeneous cylinder about an axis through its centre of mass and perpendicular to its circular end surfaces, is given by $I = \frac{1}{2}MR^2$ in which M is its mass and R its radius. (This result will be deduced in section 3.5.) A solid homogeneous solid cylinder with mass 0,2 kg, has a radius of 50 mm. A non-stretchable ribbon is fixed at a point on the perimeter and wound around it. The free end of the ribbon is fixed and the cylinder allowed to fall from rest as shown in Figure 3.4-15. Calculate the speed of the centre of mass of the cylinder and its angular velocity about its centre of mass after it has fallen a vertical distance of 4 metre. $g = 10 \text{ m s}^{-2}$.

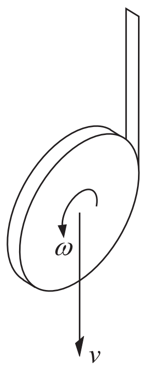


Figure 3.4-15

While the cylinder is falling, its gravitational potential energy is converted to kinetic energy of translation of the centre of mass and rotational kinetic energy about its centre of mass. Since the ribbon does not stretch or slip, the following relationship is valid:

$$\omega = v/0,05 = 20v$$

The change in gravitational potential energy is given by

$$\Delta E_p = 0,2 \times 10 \times 4 = 8 \text{ J}$$

The gain in kinetic energy is given by

$$\Delta E_k = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}(\frac{1}{2}MR^2)(v/R)^2 = 0,15v^2 \text{ J}$$

so that $0,15v^2 = 8$ and $v = 7,303 \text{ m s}^{-1}$

The angular velocity is given by

$$\omega = 7,303/0,05 = 146,1 \text{ rad s}^{-1}$$

3.5 The calculation of moments of inertia

The moment of inertia of a body about a given axis is defined by the sum in Equation 3.4(12) which is as follows:

$$I = \sum_i m_i R_i^2 \qquad 3.4(12)$$

in which m_i is the mass of point mass number i and R_i the radius of the circle on which it moves about the axis of rotation. The summation is over all the point masses of the entire body.

If a body consists of a relatively small number of particles which can be approximated as point masses, this definition may be applied directly. If the moment of inertia of a homogeneous body has to be calculated, another approach has to be followed. The body is subdivided into a large number of small mass elements, Δm_i which are small enough to be treated as point masses. The approximate moment of inertia about the given axis is given by

$$I \approx \sum_i \Delta M_i R_i^2$$

The smaller the chosen elements are, the better the approximation becomes. It becomes an equality when the elements tend to zero.

$$I = \lim_{\Delta m_i \to 0} \sum_i \Delta m_i R_i^2$$

which is the same as the following integral:

$$I = \left(\int R^2 dm \right)_{\text{over the entire body}} \qquad 3.5(1)$$

in which the integral is to be calculated *over the entire body.* The element of mass, dm, is usually expressed in terms of a density (e.g. α, μ or ρ) and one or more position co-ordinates. R, also, is written in terms of one or more position co-ordinates and then the integral may be calculated for the entire body.

The answer will always be in the form $I = M\Re^2$ in which M is the mass of the body and $\Re$, which contains one or more of the dimensions of the body, is called the **radius of gyration** of the body about the given axis. The radius of gyration is defined by the following equation:

$$\Re = (I/M)^{\frac{1}{2}} \qquad 3.5(2)$$

Example: In problem 1 the moment of inertia of a homogeneous sphere about an axis through its centre of mass is given by $I = 0,4MR^2$. In this case the radius of gyration of the sphere is given by

$$\Re = (0,4MR^2/M)^{\frac{1}{2}} = 0,6325R$$

The radius of gyration is the radius of the circular orbit which a point mass with the same mass as the body should follow in order to have the same moment of inertia as the entire body.

3.5.1 Steiner's theorem for parallel axes

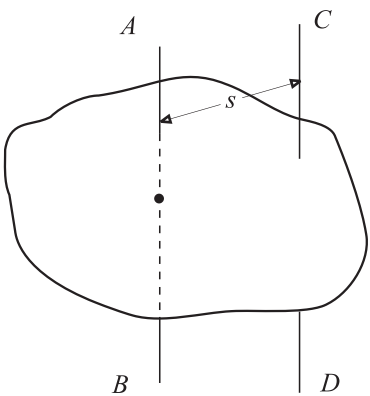


Figure 3.5-1

Before calculating the moments of inertia of a number of regular bodies, it is necessary to consider **Steiner's theorem for parallel axes**. If the moment of inertia of a body about an axis through its centre of mass is known, this theorem enables one to calculate the moment of inertia about any other axis parallel to that through the centre of mass.

Consider the body in Figure 3.5-1. AB is an axis through its centre of mass. CD is parallel to AB at distance s from it. The mass of the body is M. I_M is the moment of inertia about axis AB and I_S that about CD. Steiner's theorem for parallel axes is as follows:

$$I_S = I_M + Ms^2 \qquad 3.5(3)$$

Many proofs for this theorem exist, but the following one is probably the shortest.

If CD is fixed and the body rotates about it with angular velocity ω, the total kinetic energy relative to the inertial system of which the origin is on CD, is given by

$$E_k = \frac{1}{2} I_S \omega^2 \qquad 3.5(4)$$

By means of the result in Equation 3.4(23), the total kinetic energy may also be written as

$$E_k = \frac{1}{2} M v_m^2 + \frac{1}{2} I_M \omega^2$$

In this case, the speed of the centre of mass will be $v_m = s\omega$ and the total kinetic energy is given by

$$\begin{aligned} E_k &= \frac{1}{2} M s^s \omega^2 + \frac{1}{2} I_M \omega^2 \\ &= \frac{1}{2}(M s^2 + I_M)\omega^2 \end{aligned} \qquad 3.5(5)$$

Steiner's theorem for parallel axes follows by comparing the right-hand sides of equations 3.5(4) and 3.5(5).

3.5.2 Moments of inertia of a thin homogeneous rod

(a) About an axis perpendicular to it through its centre of mass.

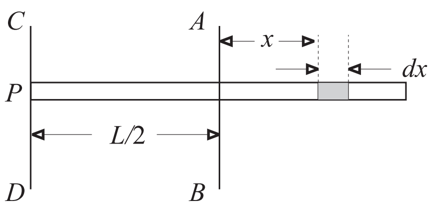


Figure 3.5-2

If the mass of the rod is M kilogram and its length L metre, its linear density (mass per unit length) is given by $\mu = M/L$. Consider an element of length, dx, at position x from the axis through the centre of mass (AB in Figure 3.5-2). The mass of this element is

$$dm = \mu dx = (M/L)dx$$

Its moment of inertia about AB is given by

$$dI = x^2 dm = x^2 (M/L) dx$$

The moment of inertia of the entire rod is the sum of that of all the elements on the rod and it may be calculated by the following integral:

$$I = \int_{-L/2}^{L/2} (M/L) x^2 dx = \frac{1}{3}(M/L) x^3 \Big|_{-L/2}^{L/2} = \frac{1}{12} M L^2 \qquad 3.5(6)$$

By means of the definition in Equation 3.5(2), the radius of gyration of the rod about any axis may be calculated. For axis AB it is $0,2887L$.

(b) About an axis perpendicular to it through one of its endpoints.

For this calculation the distance x which is shown in Figure 3.5-2, will represent the position of the element of mass relative to CD. The moment of inertia of the element is exactly the same as that in the previous calculation. The integral is also the same, but the limits of the integral are different.

$$I = \int_0^L (M/L)x^2 dx = \frac{1}{3}(M/L)x^3\Big|_0^L = \frac{1}{3}ML^2 \qquad 3.5(7)$$

About this axis, the radius of gyration is $0,5774L$. The result in Equation 3.5(7) may also be calculated by using Steiner's theorem for parallel axes.

$$I_S = I_M + Ms^2 = \frac{1}{12}ML^2 + M(L/2)^2 = \frac{1}{3}ML^2$$

3.5.3 The moment of inertia of a uniform homogeneous flat rectangular plate about an axis perpendicular to it and through its centre of mass.

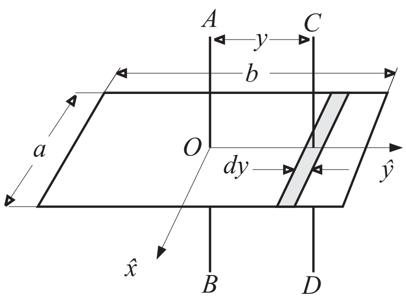


Figure 3.5-3

Figure 3.5-3 shows a rectangular plate of dimensions a metre by b metre and mass M kilogram. Since it is homogeneous and uniform, it has a constant surface density of $\alpha = M/ab$ kg m^{-2}. In terms of its surface, the mass may be written as $M = \alpha \times ab$ kg.

Choose a Cartesian frame of reference as shown in the sketch. Consider a strip at position y from the x-axis with width dy. This strip may be regarded as a thin rod of which the moment of inertia about an axis through its centre of mass (CD) which is parallel to the axis AB, is given by

$$dI_{CD} = \frac{1}{12}a^2 dm = \frac{1}{12}a^2(\alpha a\, dy)$$

By means of the theorem of parallel axes the moment of inertia of this strip about AB may be calculated.

$$dI_{AB} = \frac{1}{12}\alpha a^3 dy + y^2 dm = \frac{1}{12}\alpha a^3 dy + \alpha a y^2 dy = \alpha a(\frac{1}{12}a^2 + y^2)dy$$

To calculate the moment of inertia of the entire plate about AB, this expression needs to be integrated between the boundaries $y = -b/2$ and $y = b/2$.

$$\begin{aligned} I_{AB} &= \int_{-b/2}^{b/2} \alpha a(\frac{1}{12}a^2 + y^2)dy = \alpha a(\frac{1}{12}a^2 y + \frac{1}{3}y^3)|_{-b/2}^{b/2} \\ &= \frac{1}{12}\alpha ab(a^2 + b^2) = \frac{1}{12}M(a^2 + b^2) \end{aligned} \quad 3.5(8)$$

The radius of gyration of the plate about this axis is $0,2887(a^2 + b^2)^{\frac{1}{2}}$. If $a \ll b$, the plate reduces to a thin rod and a^2 may be disregarded. The expression is then the same as that in Equation 3.5(6). The thickness of the plate has no significance beyond the fact that uniformity was assumed.

3.5.4 The moment of inertia of a thin-walled hollow cylinder about an axis through its centre of mass perpendicular to its circular cross-section

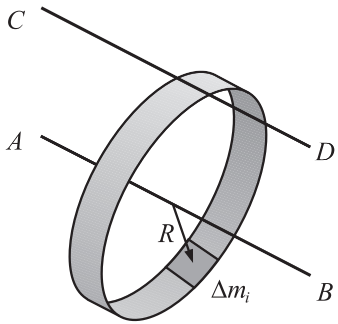


Figure 3.5-4

The specification of a thin-walled cylinder is to be interpreted that its thickness may be disregarded when compared to its radius. All the material in such a hoop will be taken to be at equal distance from the axis.

The perimeter is divided into a large number of equal segments of mass, Δm_i. Each segment is at distance R from axis AB as shown in Figure 3.5-4. R is the radius of the cylinder. The moment of inertia of segment number i is $\Delta m_i R^2$ and that of the entire cylinder

$$I = \sum_i \Delta m_i R^2 = MR^2 \quad 3.5(9)$$

The radius of gyration is equal to the radius of the cylinder. The length of the cylinder does not enter the calculation because it does not affect the mass distribution about the axis. Steiner's theorem for parallel axes may be used to calculate the moment of inertia about other axes parallel to the one above. As an exercise the reader can show that the moment of inertia about axis CD is $2MR^2$ and the radius of gyration about CD is $1,4142R$.

3.5.5 The moment of inertia of a homogeneous solid right cylinder about an axis through its centre of mass and perpendicular to its circular cross-section

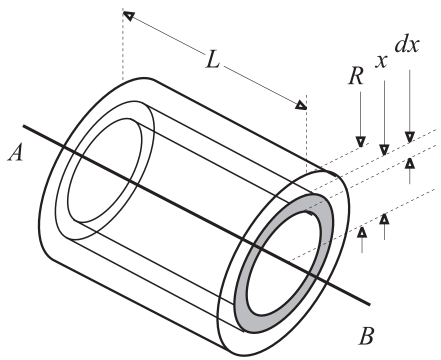


Figure 3.5-5

Consider a homogeneous cylinder of length L, radius R and mass M. Since it is homogeneous, the density is constant over the entire volume and given by

$$\rho = M/\pi R^2 L \ \text{kg m}^{-3}$$

In terms of its density, the mass of the cylinder is given by

$$M = \pi R^2 L \rho$$

Consider a hollow cylindrical element with radius x and thickness dx as shown in Figure 3.5-5. Its mass is given by

$$dm = \text{volume} \times \text{density} = (2\pi x L\, dx) \times \rho$$

Since the thickness is infinitesimal its moment of inertia is given by Equation 3.5(9).

$$dI = dm\, x^2 = 2\pi\rho L x^3\, dx$$

The moment of inertia of the entire cylinder is given by the integral between the boundaries $x = 0$ and $x = R$.

$$I = \int_0^R 2\pi\rho L x^3 dx = 2\pi\rho L(x^4/4)\Big|_0^R = \frac{1}{2}(\pi R^2 L\rho)R^2$$

The expression between brackets, is the mass, M, of the cylinder and the answer is

$$I = \frac{1}{2}MR^2 \qquad 3.5(10)$$

The radius of gyration of the cylinder about this axis is $0,7071R$.

3.5.6 The moment of inertia of a homogeneous hollow right cylinder about an axis through its centre of mass and perpendicular to its circular cross-section

The calculation for the hollow cylinder is identical to the calculation for the solid cylinder. The only difference is the lower boundary for the integral which is the radius of the co-axial hollow space, r. As an exercise the reader can confirm the result.

$$I = \frac{1}{2}M(R^2 + r^2) \qquad 3.5(11)$$

If the wall is thin ($r \approx R$), this result reduces to that given for a thin-walled cylinder in Equation 3.5(9). The radius of gyration of the cylinder about the given axis is $0,7071(R^2 + r^2)^{\frac{1}{2}}$.

3.5.7 The moment of inertia of a homogeneous solid sphere about an axis through its centre

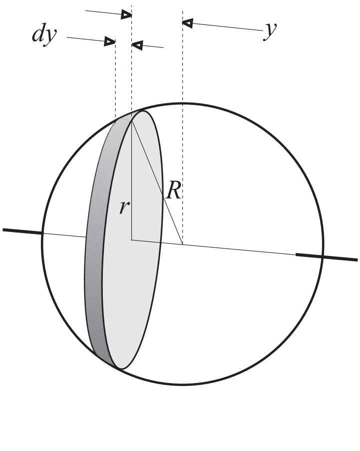


Figure 3.5-6

Consider a solid sphere with radius R metre and mass M kilogram. Since it is homogeneous, its density is constant over the entire volume and is given by

$$\rho = 3M/4\pi R^3$$

In terms of its density, the mass may be written as

$$M = \frac{4}{3}\pi R^3 \rho$$

Consider a section of the sphere perpendicular to the axis at position y from the centre and thickness dy. The radius, r, of the section depends upon the value of y. By the theorem of Pythagoras

$$r = (R^2 - y^2)^{\frac{1}{2}}$$

Since its thickness is infinitesimal, the section may be treated as a solid cylinder and its moment of inertia about axis AB is given by

$$dI = \frac{1}{2}dm\, r^2 = \frac{1}{2}(\rho\pi r^2 dy)r^2 = \frac{1}{2}\rho\pi r^4 dy = \frac{1}{2}\rho\pi(R^2 - y^2)^2 dy$$

$$dI = \frac{1}{2}\rho\pi(R^4 - 2R^2y^2 + y^4)dy$$

The moment of inertia of the entire sphere is given by the definite integral between the boundaries $y = -R$ and $y = R$.

$$\begin{aligned} I &= \int_{-R}^{R} \frac{1}{2}\pi\rho(R^4 - 2R^2y^2 + y^4)dy \\ &= \frac{1}{2}\pi\rho(R^4y - \frac{2}{3}R^2y^3 + \frac{1}{5}y^5)|_{-R}^{R} \\ &= \frac{8}{15}\pi\rho R^5 = \frac{2}{5}(\frac{4}{3}\pi R^3\rho)R^2 = \frac{2}{5}MR^2 \end{aligned} \qquad 3.5(12)$$

About an axis through its centre, the radius of gyration of a sphere is $0,6325R$.

3.6 The energy of a point mass in a circular orbit

The theory of a point mass in a circular orbit has two important applications: (i) An earth satellite in a circular orbit. (ii) The Bohr model of the hydrogen atom. In both cases the motion is governed by a central force which obeys the inverse square law. The results for the two applications will thus have similarities.

3.6.1 An earth satellite in a circular orbit

Usually earth satellites move in elliptical orbits with the earth in one of the focal points. A circle is a special case of an ellipse and it is possible for a satellite to move in a circular orbit. The circular orbit will be analysed in this section.

Consider an earth satellite of mass m kilogram and position vector $\bar{r}$ relative to the centre of mass of the earth of which the mass is M kilogram. For the purpose of this study, the earth will be approximated by a homogeneous sphere with radius R metre. Friction and other dissipative forces are disregarded. The force acting upon the satellite is

$$\bar{F} = -(GMm/r^2)\hat{r} \qquad 3.6(1)$$

in which G is the gravitational constant in Newton's law for gravitation. If the satellite moves at a constant speed of v m s^{-1} in a circular orbit with radius r metre, this force is the necessary centripetal force.

$$F = GMm/r^2 = mv^2/r$$

This gives the relationship between v and r.

$$r = GM/v^2 \qquad \text{or} \qquad v^2 = GM/r \qquad 3.6(2)$$

It is possible to place a satellite in the orbit of ones choice, but Equation 3.6(2) shows that a specific speed v exists for each circular orbit.

The satellite has kinetic energy which is given by

$$E_k = \frac{1}{2}mv^2 \qquad (\text{or}\frac{1}{2}I\omega^2 = \frac{1}{2}(mr^2)(v/r)^2 = \frac{1}{2}mv^2)$$

By means of the result in Equation 3.6(2) v may be eliminated and the kinetic energy calculated in terms of r.

$$E_k = \frac{1}{2}m(GM/r) = GMm/2r \qquad 3.6(3)$$

The kinetic energy of a satellite is thus known unambiguously if the radius of the circular orbit is specified. It is inversely proportional to the radius. This means that a satellite in a high orbit has a lower kinetic energy (and speed) than one of the same mass in a low orbit. The kinetic energy *diminishes* if the radius increases.

The potential energy of a mass in the gravitational field of the earth was calculated in chapter 2 and is given by Equation 2.5(14)

$$E_p = -GMm/r \qquad 3.6(4)$$

That the gravitational potential energy can be negative only, is the result of the choice that it was taken to be zero where $r = \infty$ and that the satellite is in a potential well because gravitation is a force of attraction. From Equation 3.6(4) it can be seen that the potential energy of the satellite will *increase* as r increases and that its maximum value is zero.

The total energy is given by

$$\begin{aligned} E &= E_k + E_p \\ &= GMm/2r - GMm/r = -GMm/2r \qquad 3.6(5) \end{aligned}$$

The total energy of a satellite on a circular orbit increases as the radius increases. If the satellite is to be gravitationally bound to the earth, the energy must remain negative. The energy becomes zero when $r \to \infty$ and then the satellite is no longer gravitationally bound to the earth. It has then escaped from the gravitational field and is free. The initial speed that an object must have on the earth's surface in order to reach this state, is called the **escape speed**.

An object may possess a positive amount of total energy but that corresponds to a free state and it will no longer move in a circular orbit. In this case the initial speed will have to be more than the escape speed. For the correct determination of the escape speed, the rotation of the earth must be taken into account. All objects which are at rest relative to the earth's surface already possess kinetic energy relative to an inertial system in which the earth's centre of mass is at rest.

Example: The mass of the earth is $M = 5,975 \times 10^{24}$ kg and its average radius, $G = 6,367 \times 10^{6}$ m. $G = 6,673 \times 10^{-11}$ $\mathrm{N\,m^2kg^{-2}}$. Disregard frictional drag and the rotation of the earth and calculate the escape speed of an object on the surface of the earth.

If the escape speed is v_0 $\mathrm{m\,s^{-1}}$, the initial kinetic energy of an object of mass m kilogram will be $E_k = \frac{1}{2}mv^2$. The potential energy on the surface of the earth is $E_p = -GMm/R$. Since the gravitational force is conservative, the total amount of energy will remain constant and for escape it will have to be equal to at least zero. The satellite will then be just free from the earth's gravitational force.

$$\begin{aligned} \frac{1}{2}mv_0^2 - GMm/R &= 0 \\ v_0 &= (2GM/R)^{\frac{1}{2}} = 1,12 \times 10^4 \ \mathrm{m\,s^{-1}} \\ &= 11,2 \ \mathrm{km\,s^{-1}} \end{aligned}$$

Equations 3.6(2) and 3.6(5) have interesting implications for a coupling manoeuvre of two space-craft which are in the same circular orbit around the earth. Consider two spacecraft A and B in the same orbit with B behind A. If B fires its engine in an effort to increase its forward velocity and catch up with A, its

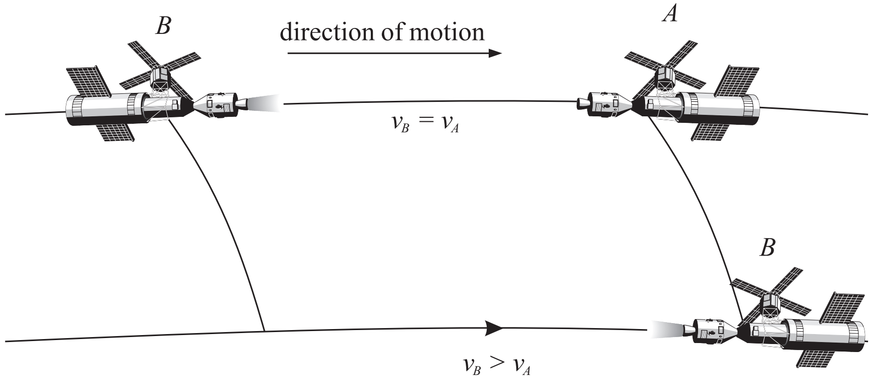


Figure 3.6-1

total energy is increased. This causes the radius of its orbit to increase in accordance with Equation 3.6(5). A larger radius corresponds to a lower speed as shown by Equation 3.6(2). The total effect is that B will rise to a higher orbit and move at a slower speed and will thus move further away from A.

If B decreases its energy by firing its retro-rockets, it moves to an orbit with a smaller radius which causes its speed to increase. It will thus pass A. Having passed A, it fires its forward engines, increases its energy and rises to the higher orbit where it will have less speed. If this is done correctly a rendezvous is possible. Figure 3.6-1 shows how this is achieved.

3.6.2 The Bohr model of the hydrogen atom

If a suitable alternating voltage is applied to a tube containing hydrogen gas at a relatively low pressure, the gas emits light which is characteristic of hydrogen. Analysis of this light by means of a spectroscope reveals a large number of pure colours of which the frequencies may be measured very accurately. This is known as an **emission spectrum**. Each element has its own characteristic emission spectrum.

In 1913 Niels Bohr succeeded in explaining the emission spectrum of hydrogen by means of a model. According to this model an atom of hydrogen consists of a massive nucleus (a proton) about which the much lighter electron may move in circular orbits. The Coulomb force is the centripetal force necessary for circular motion. The electric charge of an electron is represented by $-e$ and that of the proton, by e.For an electron, mass m, in a circular orbit of radius r at a speed of v, we may write

$$F = ke^2/r^2 = mv^2/r$$

in which k is the constant in Coulomb's law. The relationship between v and r follows from this equation.

$$r = ke^2/mv^2 \qquad \text{or} \qquad v^2 = ke^2/mr \tag{3.6(6)}$$

In exactly the same manner as for an earth satellite, the total energy of the electron in its orbit may be calculated (see section 2.2.5).

$$E = -ke^2/2r \tag{3.6(7)}$$

Up to this stage the calculations for the Bohr model and earth satellite were identical. According to these results the electron may occupy any orbit and each orbit corresponds to a definite value of the total energy. In this respect, the Bohr model is in conflict with classical theory since an accelerating electric charge always emits electromagnetic waves which diminish its energy (a body in a circular orbit is constantly accelerating). Less energy will cause the radius of orbit of the electron to decrease with a corresponding increase in speed. In fact, the electron will move on a spiral and end up in contact with the proton, the only stable configuration for the atom.

In order to explain the hydrogen spectrum in terms of this model, Bohr was forced to make two postulates which cannot be explained by classical theory. They are as follows:

1. The hydrogen atom may exist only in certain stable or steady states, each of which corresponds to a definite total energy. When the electron is in one of these orbits, it does not emit radiation even though it constantly experiences a centripetal acceleration. Transitions between orbits are possible and such transitions are accompanied by the absorption or emission of the exact amount of energy equal to the difference in the energies associated with the orbits.

2. The frequency (number of cycles per second), ν, of the radiation which is absorbed or emitted during a transition from the initial state with energy E_i to the final state with energy E_f, is given by

$$|E_i - E_f| = h\nu \qquad 3.6(8)$$

   in which $h = 6,625 \times 10^{-34}$ joule-seconds, is known as **Planck's constant**. This quantity has the same units as angular momentum.

Accurate measurements of the frequencies of the hydrogen spectrum enabled Bohr to explain the stationary states if it is assumed that the angular momentum of the electron is limited to integer multiples of $h/2\pi$. This ratio occurs so often in the study of physics that the abbreviation $\hbar = h/2\pi$ (pronounced **h-bar**) is used. Using this limitation, the angular momentum of the electron may be written as

$$J = mvr = n(h/2\pi), \qquad n = 1, 2, 3, 4 \ \ldots\ldots \qquad 3.6(9)$$

in which n is a positive integer which is called a **quantum number**. This condition, which is called the **quantisation of angular momentum**, supplies a new relationship between v and r which is independent of that shown in Equation 3.6(6). If v is eliminated from Equations 3.6(6) and 3.6(9), the following expression is obtained for the radii of the allowed orbits:

$$r_n = (h^2/4\pi^2 me^2 k)n^2 \qquad n = 1, 2, 3, 4 \ \ldots\ldots$$

The subscript n used with the radius r, indicates that an allowed radius exists for each value of the quantum number. The radii of the orbits are thus in the ratio 1:4:9:16.... If these values for the radii are substituted in 3.6(7), the equation for the energies for the stable states are obtained.

$$E_n = -(2\pi^2 k^2 m e^4/h^2)(n^{-2}) \qquad n = 1, 2, 3, 4 \ \ldots\ldots \qquad 3.6(10)$$

The lowest stable energy state in which an atom can exist, is called its **ground state**. For this state $n = 1$. The other states are called **excited states**.

According to the second postulate the energy of an emission which is the result of a transition from an initial state with quantum number n_i to a final state with a lower quantum number n_f is given by

$$\begin{aligned} E &= h\nu = E_i - E_f \\ &= (2\pi^2 k^2 m e^4/h^2)(n_f^{-2} - n_i^{-2}) && 3.6(11) \\ \text{so that} \qquad \nu &= (2\pi^2 k^2 m e^4/h^3)(n_f^{-2} - n_i^{-2}) && 3.6(12) \end{aligned}$$

The group of spectral lines which are formed by transition to the ground state, is known as the **Lyman series**. This series represents the most energetic emissions of a hydrogen atom. It is invisible to the human eye and lies in the **ultraviolet region**. The series which corresponds to transition ending on the $n = 2$ level, is known as the **Balmer series**. Series with lower energies exist and they are known as the **Paschen series**, the **Bracket series** and the **Pfund series**. An energy-level diagram for the hydrogen atom and also possible transitions are shown in Figure 3.6-2.

If the electron possesses a total energy larger that zero, it is free from the nucleus and the atom is said to be **ionised**. From Equation 3.6(10) it may be calculated that the energy of the $n = 1$ state is $-2,2 \times 10^{-18}$ J = -13,6 eV. A hydrogen atom in its ground state needs at least 13,6 eV in order to be ionised. It is also said that the **ionisation potential** of hydrogen in its ground state is 13,6 volts.

The higher the quantum number, the more densely the energy states are spaced on an energy diagram. At energy larger than $E = 0$ the electron is free and may have any energy. The energy of a free electron is not quantised and this energy range is called the **continuum energy range**.

Sommerfeld made a refinement of this model by showing that small discrepancies could be eliminated by reducing the problem to a single-particle model. Problem 30 deals with the concept of **reduced mass**. If the reduced mass of the proton-electron system, which differs only slightly from that of an electron, is substituted for m in the expressions above, the description is perfect.

In spite of the success of the Bohr model for the description of the spectrum of hydrogen (and other atoms which are ionised to the extent that they have one electron only), the theory could not be extended to include more complex atoms. The postulates do, however, largely hold their validity in **quantum mechanics** which was developed from the Bohr theory.

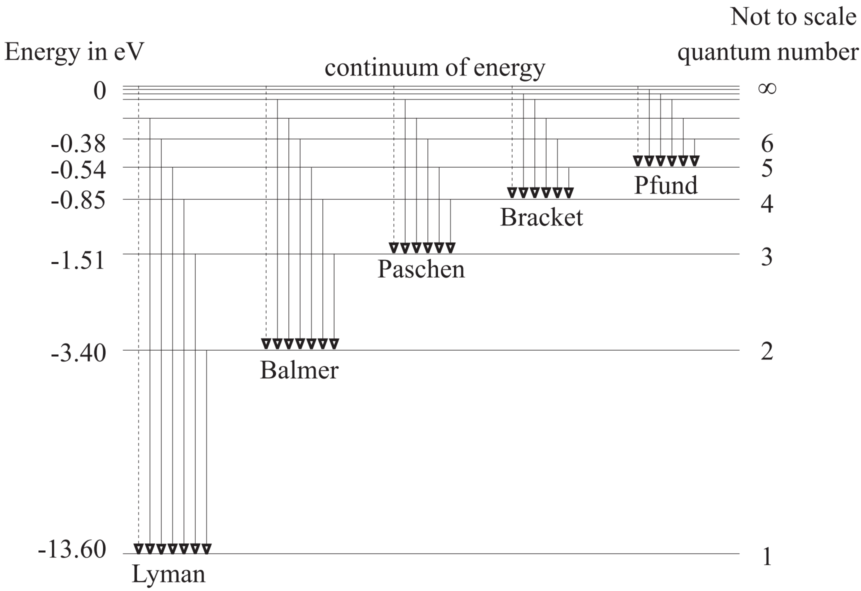


Figure 3.6-2

3.7 The gyroscope

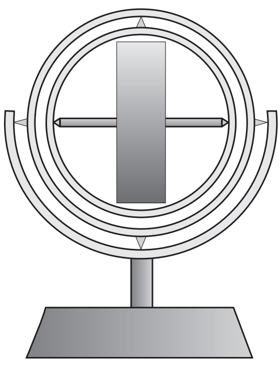


Figure 3.7-1

A gyroscope is a wheel of which the moment of inertia about its rotation axis is relatively high. To enhance its performance the friction in the bearings is kept to a minimum. Some models are mounted in **gimbals** (attributed to Cardano and thus also known as a **Cardano gimbals**) which isolates it from an external torque when the orientation of the mounting stand changes. Figure 3.7-1 shows a gyroscope mounted in gimbals. In this case the centre of mass of the wheel coincides with that of the wheel and two inner rings.

A rotating gyroscope has the following two properties:

(i) In the absence of an external torque (that is the function of the gimbals) the angular momentum will be conserved. This means that the direction of the axis will remain unchanged.
(ii) In the presence of an unbalanced external torque, the axis of the gyroscope will perform a motion known as precession.

Both these properties have many applications, some of which will be discussed in the following paragraphs.

3.7.1 The precession of a gyroscope in the presence of an unbalanced external torque

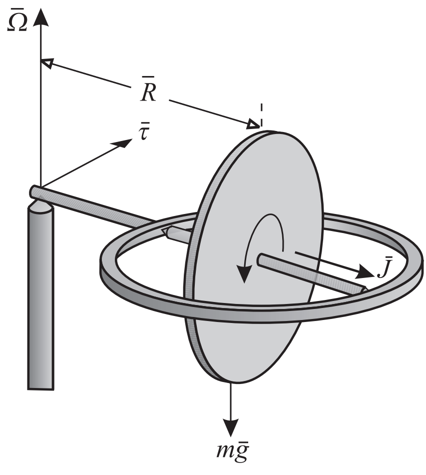


Figure 3.7-2

The theory of a body rotating about a free axis whilst experiencing an unbalanced external torque is rather complex. In order to understand a few of the properties of such a motion, the following simple model will be considered.

Figure 3.7-2 shows a gyroscope mounted in a ring. It is suspended by a mounting which is pivoted at one end. The axis of the spinning wheel is horizontal.

The angular momentum of the wheel, $\bar{J}$, is parallel to the axis and it passes through the pivot, O. In this position the weight of the system has a torque about the pivot which is given by

$$\bar{\tau} = \bar{R} \times m\bar{g}$$

In this set-up the torque vector remains perpendicular to the angular momentum vector and it will have the tendency to tilt the axis downwards. If the wheel has no angular momentum, that is exactly what will happen. If the wheel possesses angular momentum, the unsuspended end of axis will describe a horizontal circle at angular velocity $\bar{\Omega}$ while the axis remains horizontal. This circular motion is a special case of a motion which is known as **precession**.

It is fairly simple to calculate the angular velocity, $\bar{\Omega}$, of the precession since the torque is perpendicular to the angular momentum and is therefore unable to change its magnitude. The translational analogue is circular motion in which the centripetal force cannot change the speed of the moving object.

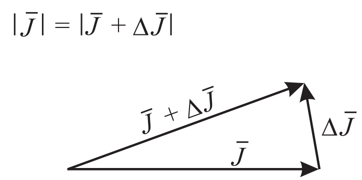


Figure 3.7-3

The direction of the angular momentum, however, must change in accordance with Equation 3.4(15) and the direction of the change must be parallel to the torque vector.

Let the change in the angular momentum vector during time interval Δt be equal to $\Delta\bar{J}$ and the corresponding angle that the horizontal axis describes, $\Delta\theta$. Figure 3.7-3 shows the vector diagram as seen from above.

For small $|\Delta\bar{J}|$, it follows from the diagram that

$$\Delta\theta = |\Delta\bar{J}|/J$$

and it follows that

$$\Omega = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t} = \lim_{\Delta t\to 0}\frac{1}{J}\frac{|\Delta\bar{J}|}{\Delta t} = \frac{1}{J}\left|\frac{\Delta\bar{J}}{\Delta t}\right|$$

By Equation 3.4(15), $|d\bar{J}/dt| = \tau$ so that the above simplifies to

$$\Omega = \tau/J \qquad \text{or} \qquad \tau = \Omega J$$

Figure 3.7-2 shows that the following vector relationship is valid:

$$\bar{\tau} = \bar{\Omega}\times\bar{J} \qquad 3.7(1)$$

This vector relationship enables one to determine the magnitude and direction of the precessional angular velocity if the angular momentum and torque vectors are known. It may also be seen that the angular velocity of the precession will increase as the angular momentum of the wheel decreases. This effect is seen clearly while observing a spinning top of which the angular velocity is decreasing. For a given angular momentum a large torque will result in a large precessional angular velocity. If the torque vector is parallel to the angular momentum vector, precession cannot occur.

The above discussion is rather over-simplified and the behaviour of the wheel will conform to the description only if the axis is initially horizontal and describing a horizontal circle at angular velocity $\bar{\Omega}$. If the unsupported end of the axis is initially supported so that $\tau = 0$, the direction of the axis will remain unaltered.

If the latter support is removed, the torque is given by $\tau = Rmg$ and that point of the axis will dip downwards. It will, however, return along an arc to its original height and the process will be repeated so that the free end of the axis follows a scalloped circular path. This deviation from a circular motion is known as **nutation**. The notation is usually damped fairly rapidly by friction and then the axis will precess with the free end pointing downwards at an angle.

The phenomena of precession and nutation are best described by means of the so-called **Euler angles** and the treatment requires a solid knowledge of vector and tensor analysis and also the use of matrix algebra. (See Goldstein: Classical Mechanics, Addison Wesley. Chapters 4 and 5). In an article by William Chase: The gyroscope: An elementary discussion of a child's toy, (*American Journal of Physics, Vol. 45, No. 11, November 1977, p 1107*) the phenomena of precession and nutation are explained in a simple way by making use of the conservation of energy and angular momentum.

3.7.2 The gyroscope in the absence of an unbalanced external torque

The behaviour of a gyroscope in the absence of torque, is a simple matter. The principle of conservation of angular momentum states that if no unbalanced external torque acts on a gyroscope, its angular momentum will be conserved. Well-constructed Cardano gimbals such as those shown in Figure 3.7-1 will limit the transfer of external torque to the spinning wheel. Unfortunately friction will prevent complete isolation and even the best gyroscopes (driven by small electric motors or air turbines) will drift from the original direction by as much as 2° per hour.

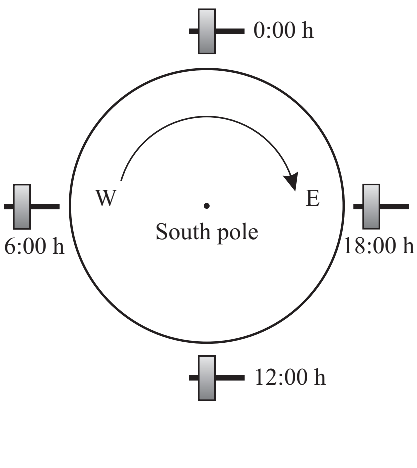


Figure 3.7-4

Figure 3.7-4 shows the orientation of a gyroscope in the equator relative to the earth over a period of 24 hours. At time 0:00 the axis is horizontal and points east-west. At 6:00 the axis is vertical with the point which was initially east, pointing upwards. At 12:00 the axis is once again horizontal but the orientation of the axis differs from what is was at 0:00 by 180°. At 18:00 it is again vertical and six hours later in the original position. It should be clear that gyroscope maintained its orientation in an inertial frame of reference while an observer on the surface of the earth made these observations.

At either pole a gyroscope with a horizontal axis will rotate once about a vertical axis in 24 hours.

3.7.3 Applications and uses of gyroscopes

When an aircraft banks or changes its angle of pitch, its magnetic compass is virtually useless. The axis of a gyroscope in a suitable suspension and which has previously been adjusted to magnetic north, will maintain its direction irrespective of the orientation of the craft. This is the principle of the **gyroscopic compass** which is standard equipment on all aircraft. Because of drift caused by friction, it unfortunately requires frequent readjustment.

Another instrument used in aircraft which makes use of the gyroscopic principle, is the **artificial horizon** by which the pilot may adjust the orientation of his craft when sight conditions are poor or during the night. The artificial horizon is operated by means of a gyroscope with a vertical axis in a weighted suspension.

In torpedoes, guided missiles and spacecraft, a system of three gyroscopes of which the axes are mutually perpendicular, is used in the principle of **inertial navigation**.

The hulls of submarines are built from steel which shields the interior from the earth's magnetic field. This renders a magnetic compass within the craft totally useless. The **gyro-compass** is an instrument which makes use of the gyroscopic principle, gravitation, the rotation of the earth and a system which damps precession effectively. It always points true north.

An extension of the gyro-compass is the **autopilot** which automatically keeps an aircraft or ship on a selected course. This system makes use of **cybernetics** in which **feedback** is used to make the necessary adjustments to the steering system when the craft drifts from the set course.

Although it is not in use in modern ships, smaller craft made use of so-called **gyro-stabilisers**. A gyro-stabiliser is a wheel with a relatively large moment of inertia about its axis of rotation and which spins at a large angular velocity within the hull of the ship. It is mounted in such a way that its precession counteracts the rolling of the ship. Modern ships have stabilising foils below the water-line which are adjusted by servo-motors to counteract the rolling motion. The servo-motors are governed by a gyroscope.

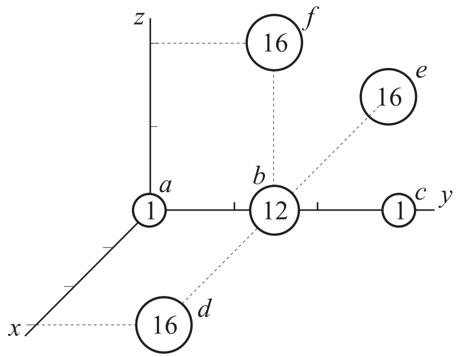
The typical shape of a bullet is chosen to minimise frictional drag while it is in flight. The sharp or round point must be in the leading position. Should the orientation of a bullet change, the frictional drag would cause it to lose speed rapidly. For this reason the barrel of a gun is fitted with helical rifling which gives the bullet a large angular momentum about its axis. The conservation of angular momentum assures that the bullet sustains its orientation throughout its flight and thus limits the effect of drag.

Many other applications of gyro-systems are in use in modern technology. Examples: Keeping the blade of a road grader at a desired level, the automatic adjusting of sights on guns which are used to fire at moving targets, conserving the orientation of a gun mounted on a moving platform, and many more. The reader is referred to the *Encyclopaedia Britannica* for a good summary of applications of gyroscopes.

3.8 PROBLEMS: CHAPTER 3

1. The mass of the earth is 6×10^{24} kg and that of the moon, $7,5 \times 10^{22}$ kg. The distance between their centres of mass is $3,8 \times 10^{8}$ m. Calculate the position of the centre of mass of the system.

2.



A hypothetical molecule consists of atoms a, b, c, d, e and f with masses 1, 12, 1, 16, 16, and 16 u respectively. Atoms a, b, c, d and e are in the plane $z = 0$. Atom f is in the $x = 0$ plane. The units along the axes are picometre (pm). The molecule is symmetrical about the planes $x = 0$ and $y = 150$ pm.

Write down the position vectors of each atom in the given frame of reference and then calculate the centre of mass of the molecule.

3. An α-particle (assume mass is 4 u) moves at velocity $2 \times 10^{8} \hat{x}\ \mathrm{m\,s^{-1}}$ along the x-axis of a Cartesian frame of reference towards a ^{12}C nucleus (assume mass is 12 u) which is at rest at the origin. At time $t = 0$ the position of the α-particle is $\bar{r}(0) = -8\hat{x}$ m. Calculate the position of the centre of mass and then its velocity.

4. A loaded rifle with mass 5 kg lies on a smooth surface (disregard friction) when it spontaneously fires a bullet with mass 5 g. Calculate how far the rifle will be from its original position when the bullet is 100 m from the position where it was fired. (Hint: Since the forces on the rifle and bullet are internal forces, the position of their centre of mass cannot change as a result of the bullet being fired.)

5. A homogeneous uniform thin wire with linear density of $\lambda\ \mathrm{kg\,m^{-1}}$, is bent into the shape of a semicircle with radius a metre. Calculate the position of its centre of mass.

6. A homogeneous uniform flat sheet which has a surface density of $\alpha\ \mathrm{kg\,m^{-2}}$, is in the shape of a semicircle with radius a metre. Calculate the position of its centre of mass.

7. Calculate the position of the centre of mass of a homogeneous solid hemisphere with radius A metre.

8. A point moves at v m s^{-1} in a circular orbit of which the radius is r metre. Calculate in each case the angular velocity, period (time for one revolution) and frequency (number of revolutions per second) if (a) $v = 4$; $r = 2$; (b) $v = 0,1$; $r = 0,04$; (c) $v = 0,2$; $r = 0,08$; (d) $v = 8\pi$; $r = 2$

9. Describe a Cartesian frame of reference on the dial of a watch with origin at the centre, $\hat{x}$ towards 3:00 and $\hat{y}$ towards 12:00. The second hand is $(6/\pi) \times 10^{-2}$ m long. (a) Calculate the speed of the tip of the second hand. (b) Calculate its velocity when it passes through the 0-mark and the 15 s-mark. (c) Calculate the change in velocity of the tip of the second hand when it moves from the 0-mark to the 15 s-mark. (d) What is the magnitude of the corresponding change in speed? (e) What is the magnitude of the corresponding change in velocity?

10. An electric motor turns at 1200 revolutions per minute and is retarded at a constant rate to 900 revolutions per minute in 4 seconds. Calculate (a) the angular acceleration, (b) the number of revolutions that it made in these 4 seconds.

11. A wheel initially turns at 1500 revolutions per minute and comes to rest while completing 100 revolutions. Calculate (a) the angular acceleration in rad s^{-2}, (b) the time taken to come to rest.

12. An object of mass 8 kg is fixed to the one end of a light cord and swung at 16 m s^{-1} in a circular orbit of radius 4 m. Calculate (a) the angular velocity of the object, (b) the tension in the cord if the weight of the object is disregarded.

13. An object is fixed to one end of a light cord of length 0,4 m and swung in a vertical circular orbit. $g = 10$ m s^{-2}. (a) Calculate the minimum speed that the object should have in the upper position so that the cord may just remain taut. (b) Calculate the speed of the object in the lowest position in order that the cord may just remain taut in the upper position.

14. An object of mass 2 kg is swung by means of a light cord to describe a vertical circular orbit of radius 4 m. $g = 10$ m s^{-2}. Calculate the tension in the cord when the object is at the highest and lowest positions of its orbit, in each case at a speed of 10 m s^{-1}.

15. A mass of 500 kg is suspended from a crane by a cable of length 15 m. A strong wind causes the load to swing, reaching an angular amplitude of 20°. (a) Calculate the vertical displacement between the equilibrium position and the turning point. (b) Use the principle of energy conservation to calculate the speed with which it passes through the equilibrium position. (c) Calculate the magnitude of the maximum value of the force that the cord must be able to withstand. At which position will that be?

16. The coefficient of kinetic friction between the tyres of a motor-car and the

road is 0,6. $g = 10\ \mathrm{m\,s^{-2}}$. Calculate the maximum speed at which the car can execute a turn on a horizontal road of which the radius of curvature is 10 m. No slipping of the tyres must occur.

17. A mass is suspended from the rear-view mirror of the motor-car in the previous question. Calculate the angle which the cord from which the object is suspended makes with the vertical when the car makes the turn at maximum speed.

18. The speed limit on a highway is 90 $\mathrm{km\,h^{-1}}$. Calculate the slope angle of the road towards the centre of curvature in order that a vehicle may complete a turn of which the radius of curvature is 200 m, at maximum speed and not make use of its steering. Assume that the road is horizontal in the direction in which the car is moving.

19. The radius of a carousel is 2,5 m. While a child sits on the perimeter, it rotates at one revolution every 4 seconds. Calculate (a) child's speed, (b) the child's acceleration, (c) the minimum static coefficient of friction between the clothing of the child and the carousel which will allow it to sit on the perimeter without holding on to anything while the motion is in progress as described.

20. A gramophone record has a radius of 150 mm. A coin lies on its surface which is horizontal. When the record is tilted, it will slip if the inclination angle exceeds 12,5°. Calculate the coefficient of static friction between the record and the coin. The normal angular velocity of the record is 33,33 revolutions per second. Determine whether the coin will be able to stay on the edge of the record while it is spinning at the correct rate. If it cannot stay on the edge at that angular velocity, calculate the position where it will be able to lie while the record rotates in this manner. If, however, it will be able to stay on the edge at the correct angular velocity, calculate the minimum angular velocity at which it will not be able to stay on the edge.

21. An **Ultracentrifuge** spins at 60 000 revolutions per minute. The magnitude of the force required to keep a particle rotating at this rate in an orbit of radius 0,1 m, is represented by F. The magnitude of the weight of the particle is W. $g = 10\ \mathrm{m\,s^{-2}}$. Calculate the ratio F/W.

22. An earth satellite (e.g. *SYNCOM*) is placed in an orbit above the equator so that it seems to be stationary above one position on earth. Calculate the radius of the orbit. Use the following data if required: Radius of the earth is $6,37 \times 10^6$ m, the mass of the earth is $5,98 \times 10^{24}$ kg, Newton gravitational constant is $6,67 \times 10^{-11}\ \mathrm{N\,m^2kg^{-2}}$, a length of a solar day is $8,640 \times 10^4$ s and the length of a sidereal day, $8,616 \times 10^4$ s.

23. An earth satellite moves in a circular orbit at a height of 300 km above the

surface or the earth. Calculate (a) its speed, (b) its period, (c) its centripetal acceleration. Use the constants given in question 22.

24. Calculate the moment of inertia of a sphere of mass 160 kg and radius 0,25 m about an axis (a) through its centre, (b) which is a tangent to the sphere, (c) which is at a distance 0,75 m from its centre.

25. A wagon wheel with eight spokes which are regularly spaced, has a radius of 0,5 m. The rim has a mass of 10 kg and each of the eight spokes (from the hub to the rim) a mass of 1,5 kg. The thickness of the rim and the spokes may be disregarded. Calculate the moment of inertia of the wheel about its axle.

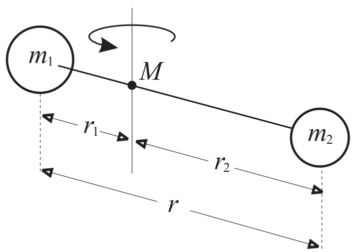
26. Calculate the moment of inertia about the x-axis of the collection of mass points shown in Figure 3.1-1.

27. Calculate the moment of inertia of the hypothetical molecule in question 2 about the following *axes*: (a) the y-axis, (b) de, (c) bf, (d) the $z-$axis.

28. A thin wedge-shaped rod AB, has a length of 2 m and its linear density depends on x, the distance from its sharp point A, and is given by $\mu(x) = 3x$ kg m^{-1}. Calculate the moment of inertia of the rod about an axis perpendicular to it through (a) A, (b) B.

29. A circular disk, radius 2 m, has a concentric hole of radius 1 m in it. The surface density of the disk is $1/(2\pi)$ kg m^{-2}. Calculate from first principles the moment of inertia of the disk about an axis perpendicular to its surface through (a) its centre of mass, (b) a point on the external perimeter (c) a point on the perimeter of the hole.

30.



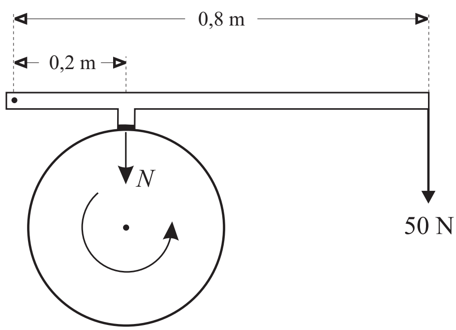
Two masses m_1 and m_2 which may be regarded as point masses for the purposes of this calculation, are at a distance r metre from each other. Mass m_1 is at distance r_1 metre and mass m_2 at distance r_2 metre from M, the centre of mass of the system. Calculate r_1 and r_2 in terms of m_1 and m_2. The system rotates about an axis through M which is perpendicular to the line joining them. Calculate the total moment of inertia of the system about this axis. Write the answer in the form $I = \mu r^2$. Write down an expression for μ in terms of m_1, m_2 and r. The quantity μ is called the **reduced mass** of the system. Instead of considering a system of two particles, it is reduced to a single particle at distance r from the axis and with mass μ. This was the correction which was needed to enable the Bohr model of the atom to describe the spectrum of hydrogen perfectly. Calculate the reduced mass of a proton-electron system. The mass of an electron is m and that of a proton $1836m$.

31. A man of mass 85 kg possesses a tape measure. He finds that a tapering straight log of which the length is 6 m, balances when pivoted is 2 m from the thick end. When he pivots it at the centre, it balances when he sits astride it at the thin end. Calculate the mass of the log. Calculate where the fulcrum should be placed to have balance when the man sits astride it on the thick end.

32. A non-uniform pole AB, has a length of 4 m and a mass of of 20 kg. The centre of mass is 1,8 m from A. John and Kevin use this pole to carry an object of mass 50 kg. It hangs 1 m from A which rests on John's shoulder. Calculate where the pole should rest on Kevin's shoulder so that he carries 0,75 of the weight that John carries.

33. To determine the centre of mass of a $3,6$ m long non-uniform straight log, a man uses two scales. The two ends of the logs are placed on the two scales. The scale on which end A rests, reads 150 kg and that on which B rests, 210 kg. Calculate the position of the centre of mass.

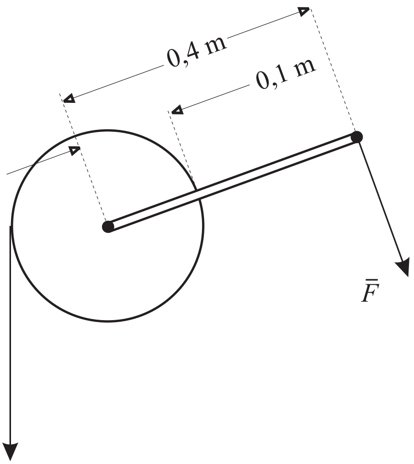
34.



The cast iron wheel in the sketch has a mass of 50 kg and a radius of 0,2 m. It is a solid cylinder and the mass of the axis may be disregarded. It is equipped with a braking system consisting of a lever of length 0,8 m and a brake shoe which is 0,2 m from the fulcrum of the lever. The coefficient of kinetic friction between the brake shoe and the cast iron is 0,4. The wheel has an angular velocity of 80 rad s^{-1} when the brake is applied by exerting a force of 50 N on the furthest end of the lever as shown in the sketch.

(a) Calculate the moment of inertia of the wheel about the rotation axis. (b) Calculate the torque of the 50 N about the fulcrum of the lever. (c) Calculate the force, N, which the brake shoe exerts on the wheel. (d) Calculate the frictional force tangentially to the rim of the wheel. (e) Calculate the torque of the frictional force about the rotational axis. (f) Calculate the angular acceleration of the wheel which is caused by the torque of the frictional force. (g) Through what angle will the wheel turn before coming to rest? (h) How long will it take to come to rest? (i) How much heat is generated during the braking process?

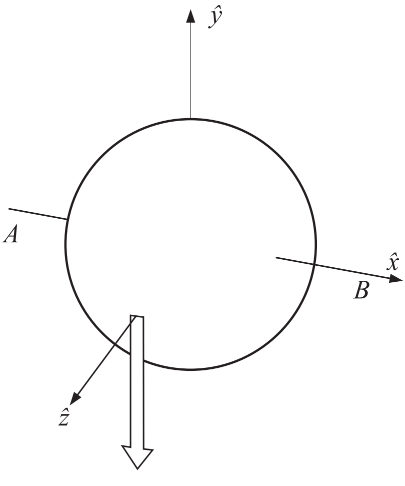
35.



The drum of a windlass has a radius of 0,1 m and the crank lever has a length of 0,4 m as shown in the sketch. It is used to hoist a 20 kg bucket of water from a 12 m deep well by means of a light cord of which the mass may be disregarded. $g = 10\ \mathrm{m\,s^{-2}}$. (a) Calculate the torque required to hoist the bucket at a constant speed. Disregard friction. (b) Calculate the magnitude of the force, F, which must be applied to the handle of the crank to supply this torque. (c) Calculate the work which is necessary to hoist the bucket to the top, in three ways: (i) using $\int \tau d\theta$, (b) using $\int \bar{F} \bullet \bar{ds}$, (iii) using energy considerations.

36. A simple bucket pump (Persian wheel) is powered by one donkey which walks on a circular track at the end of a 3 m long lever. It completes one revolution each 8 seconds. The force which the animal exerts on the end of the lever is 250 N tangential to the track. Calculate (a) the angular velocity of the donkey in $\mathrm{rad\,s^{-1}}$, (b) the torque which the donkey exerts on the system, (c) the power of the donkey, (d) the stream of water delivered by the system which is 25% efficient. The depth of the well is 5 m, the density of water is $10^3\ \mathrm{kg\,m^{-3}}$ and $g = 10\ \mathrm{m\,s^{-2}}$.

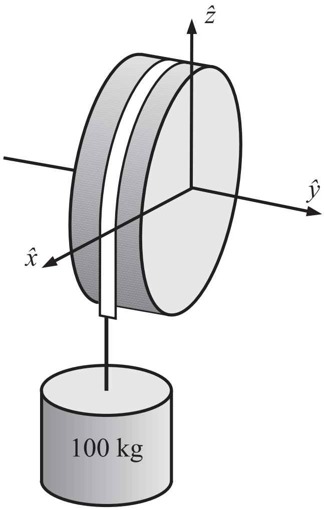
37.



The sphere in the sketch has a radius of 0,25 m and a mass of 160 kg. It is able to rotate about a light axis AB of which the friction may be disregarded. A constant tangential force of $-48\hat{y}$ N acts on the perimeter of the sphere. Its initial angular velocity is $\bar{\omega}\hat{x} = -15\hat{x}$ $\mathrm{rad\,s^{-1}}$. (a) Calculate the moment of inertia of the sphere about the axis of rotation, AB. (b) Calculate the torque which acts on the sphere. (c) Calculate its angular acceleration. (d) Calculate the angular velocity as a function of time. (e) Calculate the angle of rotation as a function of time. Assume that $\theta(0) = 0$. (f) When will the sphere be at rest? (g) What is the sphere's kinetic energy at $t = 5$ s and $t = 7$ s?

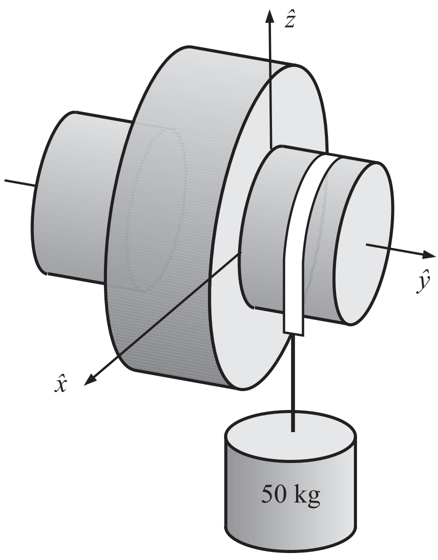
(h) How much work does the torque do between $t = 5$ s and $t = 7$ s?

38.



The solid homogeneous cylindrical wheel in the sketch has a mass of 50 kg and a radius of 0,2 m and it can rotate about a light axis of which the friction may be disregarded. A long unstretchable ribbon is wound around it. At the free end of the ribbon a mass of 100 kg is suspended. The other end of the ribbon is fixed to the wheel so that it will not slip. The 100 kg is allowed to move downwards from rest so that it sets the wheel in motion. $g = 10\ \mathrm{m\,s^{-2}}$. Calculate (a) the moment of inertia of the wheel about the rotational axis, (b) the final angular velocity of the wheel if the 100 kg moves 4 m vertically downwards, (c) the angle through which the wheel turns during this process, (d) the angular acceleration of the wheel, (e) the torque which accelerates the wheel, (f) the tension in the ribbon while the wheel accelerates, (g) the tension in the ribbon if the wheel is held at rest before the motion is allowed to start.

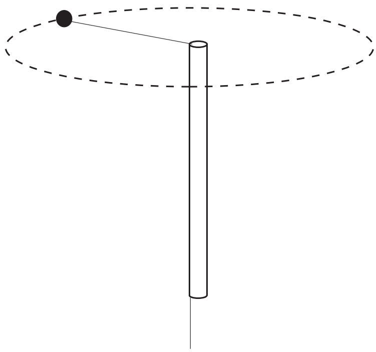
39.



The solid cylinder of 200 kg has a radius of 0,4 m. The two identical portions of the axle which are coaxially fixed to it on opposite sides each have a mass of 50 kg and radius 0,2 m. Friction may be disregarded. A long thin light unstretchable ribbon is wound around one side of the axle as shown in the sketch. A mass of 50 kg is suspended from the free end and the other end is firmly fixed to the axle so that it will not slip. Calculate (a) the moment of inertia of the wheel and axle about the axis of rotation, (b) the angular velocity of the cylinder after the 50 kg has descended a vertical distance of 10 m from rest, (c) its angular acceleration, (d) the torque exerted on the wheel while it moves, (e) the acceleration of the 50 kg, (f) the tension in the ribbon if the wheel is held at rest before the motion is allowed to begin and also after the motion has commenced, (g) the resultant force on the 50 kg while the motion is in progress.

40. The public transport system of a city works as follows: Each bus is fitted with an electric motor and a heavy flywheel. At each stop electrical power is available to supply rotational kinetic energy to the flywheel by means of the electric motor. The kinetic energy of the flywheel is used to drive the bus against friction to the following stop. The mass of the solid cylindrical flywheel is 500 kg and its radius 1 m. The mass of the bus and its contents is 5000 kg and all the frictional forces are equivalent to a coefficient of kinetic friction of 0,01. (This assumption is not realistic since frictional drag will be dependent on the speed. The assumption is made to simplify the problem.) $g = 10\ \mathrm{m\,s^{-2}}$. Calculate the maximum distance that a bus will reach if the initial angular velocity of the flywheel is $6000/\pi$ revolutions per minute.

41.

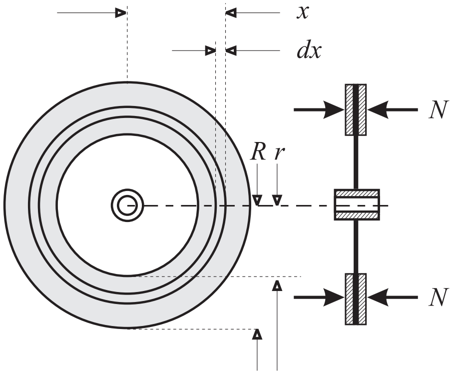


The sketch shows a mass of m kg which is swung in a horizontal circular orbit of radius r metre at speed $v\ \mathrm{m\,s^{-1}}$. The light cord passes through a thin tube and the lower end is hand-held while the mass is in motion. Disregard the weight of the mass, i.e. think if the experiment as if it were performed in outer space. While the portion of the cord which is fixed to the mass is at right angles to the tube, the lower end is pulled downwards sharply until the radius of the motion is $0{,}5r$. Calculate the new speed of the mass.

42. A hoop, a solid sphere and a solid cylinder have the same mass and radius. They are placed in a row at the top of an inclined plane and allowed to roll down from rest without slipping. Calculate the speed of each when it reaches the end of the inclined plane and determine the order in which they will arrive if they start simultaneously.

43. An engineer wishes to measure the power of an internal-combustion engine experimentally. For this purpose he mounts the flywheel used in problem 34 , on the axle of the engine. The same braking system is used. The lever is horizontal and the torque is varied by suspending different weights from the free end. $g = 10\ \mathrm{m\,s^{-2}}$. It is found that the engine is able to sustain an angular velocity of 3000 revolutions per minute with a mass of 25 kg at the end of the lever. Calculate the power of the engine at this angular velocity.

44.



One type of clutch used in motor vehicles consists essentially of a circular steel plate which fits onto the end of the axle of the gearbox. Two rings of heat-resistant material (similar to that used for brake shoes) are fitted on either side of the plate as shown in the sketch. The clutch-plate is pressed against the flywheel by means of a pressure plate. The force is supplied by stiff springs. The friction between the clutch-plate and the combination of flywheel and pressure plate which are fixed to one another, actuates the coupling.

The problem is to calculate the maximum power which can be transferred through a given clutch-plate from the engine to the wheels of the vehicle. The calculation might seem somewhat roundabout but it is quite simple. To help the reader it is calculated in a number of simple steps. Consider a clutch-plate with inner radius r and outer radius R. The normal force between the pressure plate and the flywheel has a magnitude of N. (a) Calculate the pressure on the ring of the clutch-plate in terms of R, r and N. (b) Consider an annulus with radius x and width dx on the clutch-plate. Calculate the normal force, dN, which acts on this annulus. (c) The coefficient of static friction between the clutch-plate and the flywheel (and also the pressure plate) is equal to μ. Calculate the frictional force, dF, on this annulus. (d) Calculate $d\tau$, the magnitude of the torque of dF about the rotational axis. (e) Calculate the total torque of the frictional force about the axis. Note that the clutch-plate has *two* sides. (f) The maximum torque which the clutch plate can exert before slipping is now known. Calculate the maximum power which may be transmitted through the system at an angular velocity ω.

45. In the evolution of a star it may be that its inward gravitational force is large enough to transform it into a **neutron star** which has a density of about 10^8 $\text{kg}\,\text{mm}^{-3}$. Although it is physically impossible, the earth would have a radius of about 100 m if it could turn into a neutron-star. (See*National Geographic, Vol 145, No 5, May 1974, p 618).* The average radius of the earth is $6,37 \times 10^6$ m. Calculate the angular velocity of the earth if it could change to a neutron star in the absence of external torque. Disregard the possibility that the forces which bind the material of a neutron star may not be of sufficient magnitude to keep the material of the earth together under these circumstances.

46. The average distance between the earth and the sun is $1,49 \times 10^{11}$ m and the mass of the sun is $1,99 \times 10^{30}$ kg. $G = 6,67 \times 10^{-11}\ \text{N}\,\text{m}^2\text{kg}^{-2}$. Disregard

the rotation and the gravitational effects of the earth and calculate the escape speed from the solar system if the object is launched from the earth.

47. The constant in Coulomb's law is $8,987 \times 10^9$ $\mathrm{N\,m^2C^{-2}}$, the magnitude of the electron charge is $e = 1,602 \times 10^{-19}$ C, the electron mass, $m = 9,108 \times 10^{-31}$ kg and Planck's constant, $h = 6,625 \times 10^{-34}$ J s. Calculate the energy of the $n = 1$ and $n = 2$ states of the hydrogen atom. Calculate the frequency of the photon emitted when a hydrogen atom in the first excited state reverts to the ground state. Also calculate the ionisation potential of hydrogen in the first excited state.

Chapter 4

PERIODIC PHENOMENA

4.1 Mathematical introduction

Before periodic phenomena are studied, it is necessary to to revise some mathematical techniques which have been used previously and to present them in a more general context.

4.1.1 Differential equations and their solution

Consider the following simple kinematic problem: The motion of a point mass is limited to the x-axis of a Cartesian frame of reference and it experiences an acceleration of $\bar{a} = 12\hat{x}\ \mathrm{m\,s^{-2}}$. Calculate the velocity and the position of the mass as functions of time.

Since it is a one-dimensional problem, vector notation may be omitted. The problem is of the kind treated as number 3 in Table 1.5.4. The solution follows directly by integration.

$$\begin{aligned} v &= \int a\,dt = \int 12\,dt = 12t + k\ \mathrm{m\,s^{-1}} && \text{with } k = v(0) \\ x &= \int v\,dt = \int (12t+k)dt = 6t^2 + kt + c\ \mathrm{m} && \text{with } c = x(0) \end{aligned}$$

Although the constants are indicated as $v(0)$ and $x(0)$, they may be calculated from known numerical values for v and x at any instant.

This problem may be rephrased in a way which accentuates the mathematical properties rather than the physical interpretation.

Given: $a = dv/dt = d^2x/dt^2 = 12$. The equation $d^2x/dt^2 = 12$ is known as a **differential equation** because it contains a derivative and which is of the **second order** because the highest order derivative which occurs is a second-order derivative.

Required: $x = x(t)$, i.e. a function of t is to be determined (the dependent variable is x) which satisfies the initial conditions $x(0) = c$ and $(dx/dt)_{t=0} = k$ and of which the second derivative to t is equal to 12. It is said that *the second-order differential equation* $d^2x/dt^2 = 12$ *is to be solved for the given initial conditions*. The function $x = x(t)$ which satisfies these conditions is known as the **solution** of the differential equation for the initial conditions.

In the example which was presented, the solution could be found simply by integrating twice:

$$d^2x/dt^2 = 12 \quad \text{so that} \quad dx/dt = \int (d^2x/dt^2)dt = \int 12\,dt$$
$$\text{and} \quad x = \int (dx/dt)dt = \int \left(\int 12\,dt\right) dt$$

in which an integration constant appears in each integration. The integration constants are calculated from the initial conditions.

The fact that this differential equation could be solved by two successive integrations is due to its fairly simple properties. Not all differential equations may be solved directly by integration. The study which deals with the solution of differential equations is fairly extensive and usually forms an important part of college and university mathematics. At this stage it is sufficient for the reader to take note of the concept of a **differential equation** and know what is meant by its **solution**.

Irrespective of the way in which a solution is determined, the method remains equivalent to an integration process and each reduction of the order by one, will produce an unknown constant which has to be calculated from the initial conditions.

Many problems in physics and engineering consist of the setting up of differential equations and the determination of their solutions for specific conditions. Some differential equations occur more often than others and it is worthwhile to study their solutions extensively.

4.1.2 A differential equation which occurs frequently in the study of periodic phenomena

A **periodic phenomenon** is one which is repeated in the same manner as time progresses. Examples: The motion of the balance wheel of a watch, the motion of a pendulum, the vertical motion of a mass hanging from the end of a helical spring, the tides at sea, the rotation of a wheel, etc.

In the study of these phenomena a differential equation of the following form occurs frequently:

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad 4.1(1)$$

in which ω^2 is a constant. At this stage it might seem strange that one would prefer to use ω-*squared* to indicate a constant. The advantages of this choice will be evident when the solutions of the equation are studied and a physical interpretation of ω is given.

As was the case with the differential equation in section 4.1.1, we may attempt to solve this one by integration. As before

$$d^2x/dt^2 = -\omega^2 x \qquad \text{so that} \qquad dx/dt = \int (d^2x/dt^2)dt = \int(-\omega^2 x)dt$$

The last term shows that integration is impossible since the solution, $x = x(t)$, has to be known before integration can be attempted. (The answer is needed to calculate the answer!) The only method remaining is to make use of knowledge about functions which have the property that if they are differentiated twice, the answer will be the original function multiplied by $-\omega^2$.

From experience with trigonometric and exponential functions, it is known that

$$\begin{aligned}
(d^2/dt^2)(\sin\omega t) &= (d/dt)(\omega\cos\omega t) = -\omega^2\sin\omega t \\
(d^2/dt^2)(\cos\omega t) &= (d/dt)(-\omega\sin\omega t) = -\omega^2\cos\omega t \\
(d^2/dt^2)(e^{i\omega t}) &= (d/dt)(i\omega e^{i\omega t}) = i^2\omega^2 e^{i\omega t} = -\omega^2 e^{i\omega t}
\end{aligned}$$

In the last function, the **imaginary quantity** $i = \sqrt{-1}$ which has the property $i^2 = -1$, was used.

By simply differentiating twice in each case, the reader may verify that each of the following functions is a solution of the differential equation given in Equation 4.1(1):

$$x = A\sin(\omega t + \alpha) \qquad A \text{ and } \alpha \text{ constant} \qquad 4.1(2)$$

$x = B\cos(\omega t + \beta)$	B and β constant	4.1(3)
$x = A\sin(\omega t + \alpha) \pm B\cos(\omega t + \beta)$	A, B, α and β constant	4.1(4)
$x = Ae^{\pm i(\omega t + \phi)}$	A and ϕ constant	4.1(5)

The quantities A, B, α, β and ϕ in these solutions are equivalent to integration constants and each has to be calculated from the given boundary conditions.

Comments:

1. The solutions of this differential equation are all **oscillatory** with a **period** of $T = 2\pi/\omega$. This means that the function value at any value of t will be the same if t increases by an amount of $\Delta t = T = 2\pi/\omega$. During this period, x is said to complete **one cycle**. The relationship between T and ω is of prime importance.

$$T = 2\pi/\omega \qquad \text{or} \qquad \omega = 2\pi/T \qquad 4.1(6)$$

2. From the second equation it can be seen that ω has the same dimensions as angular velocity, i.e. radians per second. This fact is of importance in the physical interpretation of ω.

3. The number of cycles completed per second, is known as the **frequency**. Frequency is measured in **cycles per second** which, in short, is called **hertz (Hz)**. Frequency is usually indicated by either f or the Greek letter ν. In this book, ν will be used. It should be clear that the period and the frequency are reciprocals of each other.

$$T = 1/\nu = 2\pi/\omega \qquad \text{or} \qquad \nu = 1/T = \omega/2\pi \qquad 4.1(7)$$

   The quantity ω is simply the frequency multiplied by 2π.

$$\omega = 2\pi\nu \qquad 4.1(8)$$

   For this reason ω is called the **angular frequency**.

4. The constants A and B in the solutions shown in Equations 4.1(2), 4.1(3) and 4.1(5), are known as the **amplitude** of the functions and each constant is the maximum value of $|x|$. The differential equation 4.1(1) contains no information about the value of the amplitude. For its calculation suitable boundary conditions are required.

5. The quantity which appears between brackets in all the solutions, e.g. $(\omega t + \phi)$, is called the **phase angle** or, in short, **phase**. The constant portion of the phase, e.g. ϕ, is known as the **phase constant** or **initial**

phase. The phase constant is the value of the phase angle when $t = 0$. As is the case with the amplitude, the differential equation contains no information about the phase constant and a suitable boundary condition is necessary to calculate it. The specification of $x(0)$ and the initial value of dx/dt are sufficient to calculate both the amplitude and the phase constant.

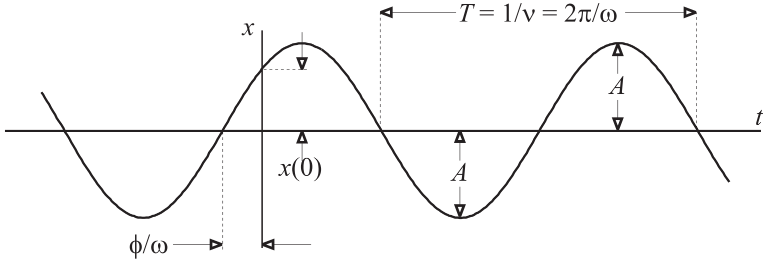


Figure 4.1-1

6. Figure 4.1-1 shows a graphical representation of the solution $x = A \sin(\omega t + \phi)$. In this figure, most of the quantities which are defined in Comments (1) to (5) are indicated. Note that $x(0) = A \sin\phi$ and that the same graph may just as well have been described by means of a cosine function with a different phase constant.

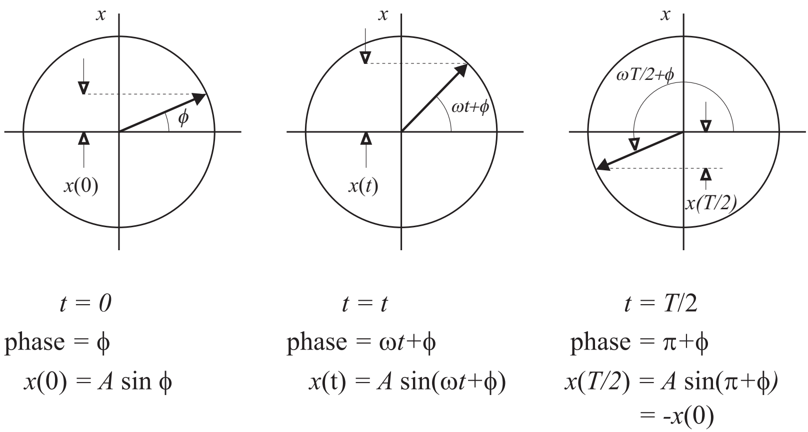


Figure 4.1-2

7. The use of a **phasor diagram** provides a simple graphical way to generate numerical values of the oscillatory solution. A **phasor** is a vector of which the length is equal to the amplitude of the function (A in Figure 4.1-1)

and with its initial point at the zero point of a scale along which x is indicated. At time $t = 0$, the phasor makes an angle which is equal to the phase constant (ϕ in Figure 4.1-2) with the direction perpendicular to the axis along which the function is plotted. The phasor rotates at a constant angular velocity ω in a positive sense (counter-clockwise). The projection of the phasor on the function-axis gives the numerical value of x at any instant t. It is said that the circular motion of the phasor and the oscillation of the function are **in harmony**. For each revolution completed by the phasor, the function completes one cycle. The physical interpretation of ω should be clear. It is the angular velocity of that phasor which is in harmony with the function.

Figure 4.1-2 shows a phasor at time $t = 0$, at time t seconds later and again at time $t = T/2 = \pi/\omega$.

The technique of phasor diagrams may be used for the composition ("addition") of two or more oscillatory phenomena of the same kind. In the study of physical optics, waves in general, and especially the theory of alternating electric currents and voltages, the use of phasors simplifies the calculations to a great extent. Phasors also form the base of the description of alternating electrical currents by means of complex numbers, i.e. numbers which contain real and imaginary portions.

4.1.3 The construction of solutions for the differential equation $d^2x/dt^2 = -\omega^2 x$

By means of an example, it will be shown how solutions for this differential equation may be constructed for a variety of boundary conditions. In this example the quantity x which is dependent on the time, t, will be considered. To keep it general, no units will be assigned to x. It should, however, be kept in mind that x could represent a position of which the units could be metres, or a velocity, an acceleration, the intensity of an electric field, an electric potential difference, or an angle, etc., in each case with the appropriate units.

Example: A physical quantity, x, depends on the time, t. The magnitude of x is determined by means of the following differential equation:

$$d^2x/dt^2 = -16\pi^2 x$$

in which the time is measured in seconds. The amplitude of the solution is 100 units. (a) Calculate the period and the frequency of the solution. (b) Construct solutions for the following initial values: (i) $x(0) = 100$, (ii) $x(0) = -100$, (iii) $x(0) = 0$ and $dx/dt > 0$, i.e. x is increasing at the instant when $x = 0$, (iv) $x(0) = 0$ and $dx/dt < 0$, i.e. x is decreasing when $x = 0$, (v) $x = 50$ and

$dx/dt > 0$, i.e. x is increasing when $x = 50$, (vi) $x(0) = 50$ and $dx/dt < 0$, i.e. x is decreasing when $x = 50$.

(a) It is not necessary to construct a solution for the differential equation to answer this portion of the question. From the differential equation it follows directly that

$$\begin{aligned} \omega^2 = 16\pi^2 \quad &\text{from which follows:} & \omega &= 4\pi \ \text{rad}\,\text{s}^{-1} \\ &\text{so that the period is given by} & T &= 2\pi/\omega = 0,5 \ \text{seconds} \\ &\text{and the frequency by} & \nu &= 1/T = 2,0 \ \text{hertz} \end{aligned}$$

This calculation emphasises the fact that the period and the frequency have no connection with the amplitude and that they are not influenced by the boundary conditions which are used for the construction of special solutions.

(b) From the discussion of the theory it is known that the solutions are oscillatory, i.e. the phenomenon is repetitive in a way shown in Figure 4.1-1. An observer who wishes to describe this phenomenon as a function of time, will have to start a stop-watch in order to define the instant $t = 0$, i.e. the zero-point on the time scale. The construction of a solution which is in accordance with given boundary conditions (initial conditions) is an exercise to allow for the possibility that an observer may choose $t = 0$ at any stage of a cycle. Different functions which correspond to different initial conditions, describe the same physical phenomenon in spite of differences in the mathematical expressions.

In order to construct a solution for the given differential equation, *any* of the functions 4.1(2) to 4.1(5) may be used. We simply *choose* the following solution:

$$x = A \sin(\omega t + \phi)$$

which will be adapted to the given differential equation and in each case to the given initial conditions.

The value of ω follows from the differential equation as shown in the answer to question (a). $\omega = 4\pi$ rad s^{-1}. The differential equation does not provide any information about the amplitude, A. In the absence of supplementary information, its value is not known. In this problem it is known that the amplitude is 100 units. Using this information, it may be written that

$$x = 100 \sin(4\pi t + \phi) \qquad \ldots\ldots(1)$$

From this the value of $x(0)$ may be calculated in terms of ϕ.

$$x(0) = 100 \sin(4\pi \times 0 + \phi) = 100 \sin \phi \qquad \ldots\ldots(2)$$

From the solution it can also be seen that

$$dx/dt = 4\pi \times 100 \cos(4\pi t + \phi)$$

$$\text{so that } (dx/dt)_{t=0} = 400\pi\cos\phi \qquad \ldots\ldots(3)$$

In each of the six examples which follow, the observer starts his stop-watch at a different stage of the cycle. It will be shown how the value of ϕ is calculated in each case.

(i) $x(0) = 100$. This means that the stop-watch is started when $x = 100$. This value is equated to the value of $x(0)$ expressed in terms of ϕ, i.e. the value in Equation(2) above.

$$\begin{aligned} 100 &= 100\sin\phi \qquad \text{from which follows} \qquad \sin\phi = 1 \\ \text{so that } \phi &= \pi/2 \pm n\times 2\pi \qquad\qquad n = 0, 1, 2, 3, \ldots \end{aligned}$$

Any of the above values for ϕ would be an acceptable phase constant. *It is, however, customary (but not necessary) to assign the smallest possible value to ϕ when the problem involves a single oscillatory phenomenon.* In accordance with this custom (which is introduced to avoid the unnecessary use of large numbers) the value $\phi = \pi/2$ is accepted. By substituting this value in Equation (1), the required solution is complete.

$$x = x(t) = 100\sin(4\pi t + \pi/2) = 100\cos 4\pi t$$

If the choice was made at the beginning rather to use a cosine function, the appropriate phase constant would have been equal to zero and the solution exactly the same. A graph of the solution is shown in Figure 4.1-3(a).

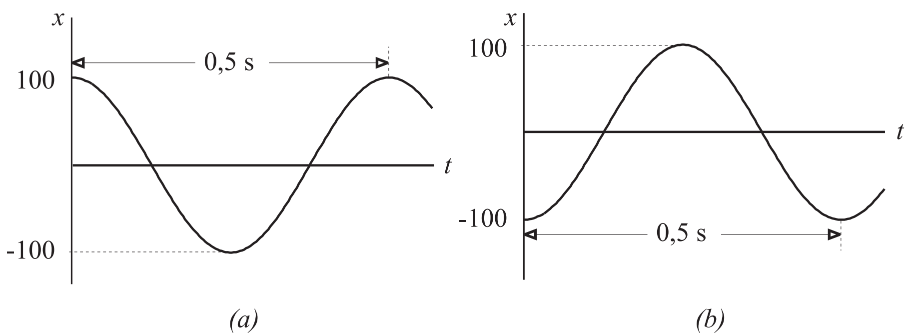


Figure 4.1-3

(ii) $x(0) = -100$. This means that the observer starts the stop-watch when $x = -100$. If this value is substituted in Equation (2), the phase constant may be calculated in exactly the same way as in the problem (i). The complete solution is as follows:

$$x = x(t) = 100\sin(4\pi t + 3\pi/2) = 100\cos(4\pi t + \pi)$$

The graph representing this solution is shown in Figure 4.1-3(b).

(iii) $x(0) = 0$ and $dx/dt > 0$. A function oscillating about $x = 0$ can be zero in two different ways – while it is increasing and while it is decreasing. In Figure 4.1-1 these two possibilities correspond to positions where the function intersects the t-axis with a positive and a negative gradient, respectively. In the previous two solutions it was not necessary to specify dx/dt since the specified initial values of x can be either a maximum (100) or a minimum (-100) in a unique way.

$$\begin{aligned} \text{As before:} \qquad x(0) &= 0 = 100\sin\phi \\ \text{so that} \qquad \sin\phi &= 0 \\ \text{and} \qquad \phi &= \pm n\pi \qquad n = 0, 1, 2, 3, \ldots \end{aligned}$$

The requirement that $dx/dt > 0$ excludes some of the values of n which are mentioned above. Only the even values of n, give positive values of dx/dt and therefore the odd values are excluded. ($dx/dt = 400\pi\cos(\text{even}n \times \pi) > 0$, and $dx/dt = 400\pi\cos(\text{odd}n \times \pi) < 0$.)

If, as before, the smallest value of ϕ is chosen, the complete solution is as follows:

$$x = x(t) = 100\sin 4\pi t$$

A graphical representation of this solution is shown in Figure 4.1-4(a). This solution is one of the simplest that can be used for a periodic phenomenon. Excluding cases in which the boundary conditions are of such a nature that the phase constant cannot be zero, one simply chooses the initial conditions to give $\phi = 0$. In practice the zero-point on the time scale is chosen when the function is equal to zero while it is increasing.

(iv) $x(0) = 0$ with $dx/dt < 0$. In this case the observer starts the stop-watch when $x = 0$ while x is decreasing. By using the same argument as in example number (iii), it follows that the solution is the same but this time with the exclusion of the even values of n. Choosing the smallest possible value of ϕ, the complete solution is as follows:

$$\begin{aligned} x = x(t) &= 100\sin(4\pi t + \pi) \\ &= -100\sin(4\pi t) \\ &= 100\cos(4\pi t + \pi/2) \end{aligned}$$

The graph which represents this function, is the mirror image about the t-axis of that shown in Figure 4.1-4(a).

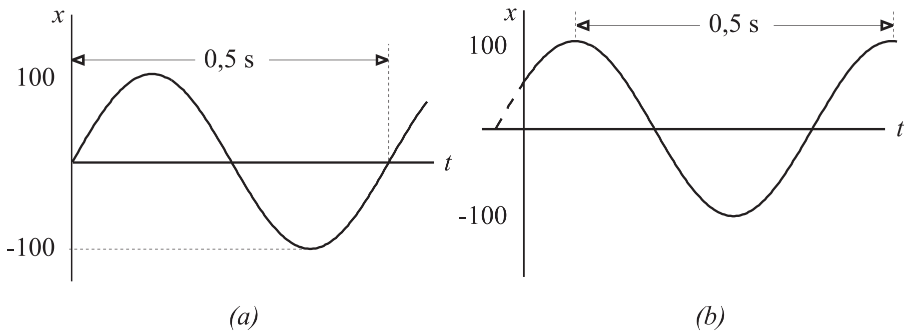


Figure 4.1-4

(v) $x(0) = 50$ with $dx/dt > 0$. This means that the observer starts the stop-watch when $x = 50$ and increasing.

$$\begin{array}{llcl} \text{As before:} & x(0) & = & 50 = 100\sin\phi \\ \text{so that} & \sin\phi & = & 0,5 \\ \text{and} & \phi & = & (-1)^n(\pi/6) \pm n\pi \qquad n = 0, 1, 2, 3, \ldots \end{array}$$

On condition that dx/dt is positive, only even values of n are admissible. The simplest complete solution is as follows:

$$x = x(t) = 100\sin(4\pi t + \pi/6)$$

The observer starts the stop-watch when the phase is equal to $\pi/6$. That means that one-twelfth of a period elapses after the function is equal to zero and increasing.

A graph which represents this solution is shown in Figure 4.1-4(b)

(vi) $x(0) = 50$ with $dx/dt < 0$. The procedure is the same as that in the previous problem. The answer is the same in all respects but for the fact that only odd values of n are admissible. With the smallest positive value of ϕ, the solution is as follows:

$$x = x(t) = 100\sin(4\pi t + 5\pi/6)$$

The observer sets the stop-watch in motion at the instant when the phase angle is equal to $5\pi/6$. When t is chosen as zero, five twelfths of a period has elapsed since $x = 0$ and decreasing.

A graph which represents this solution, is shown in Figure 4.1-5.

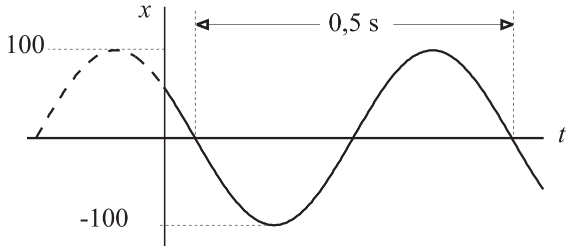


Figure 4.1-5

4.2 Linear oscillations

4.2.1 Definitions and properties of linear harmonic motion

Consider a particle which is in **stable equilibrium** under the action of a constraining force (one which limits its motion to a given space curve) and a conservative force. The origin is chosen at the position where the body is in equilibrium. At the origin the resultant of the forces on the particle is thus equal to zero.

When the particle is moved away from this position, the resultant force will not be equal to zero and will, by the nature of the equilibrium, tend to bring it back to the origin. The direction of the resultant force is thus opposite to the displacement and is called a **restoring force**.

If the particle is constrained to motion along the x-axis, the force is given by

$$F = -kx \qquad \text{or} \qquad F + kx = 0 \qquad 4.2(1)$$

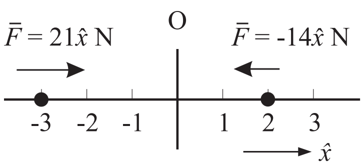


Figure 4.2-1

in which the proportionality constant, k, is known as the **force constant** or the **stiffness**.
If the particle is in motion under the action of this force, it executes **simple harmonic motion** (abbreviation: SHM) or **simple periodic motion** (SPM). If the mass of the particle is m, it follows from Newton's second law that:

$$F = ma = m(d^2x/dt^2)$$

If this expression for F is substituted in equation 4.2(1), the differential equation for the motion is obtained.

$$d^2x/dt^2 = -(k/m)x \qquad \text{or} \qquad d^2x/dt^2 + (k/m)x = 0 \qquad 4.2(2)$$

This differential equation has the same form as that which was studied in sections 4.1.2 and 4.1.3. Without even considering a solution, the following may be calculated:

$$\omega^2 = k/m \qquad \text{so that} \qquad \omega = (k/m)^{1/2} \qquad 4.2(3)$$

From the relationships $\omega = 2\pi\nu = 2\pi/T$ it follows that

$$T = 2\pi(m/k)^{1/2} \qquad \text{and} \qquad \nu = (1/2\pi)(k/m)^{1/2} \qquad 4.2(4)$$

For the initial conditions $x(0) = 0$ and $v(0) > 0$, the solution of the differential equation is as follows:

$$x = x(t) = A \sin \omega t = A \sin [(k/m)^{1/2} t] \qquad 4.2(5)$$

The amplitude, A, is determined by the initial disturbance from the position of stable equilibrium. *It is noteworthy that the amplitude has no connection with the period.* So, for example, if two identical masses are suspended from identical light elastic helical springs in a vacuum, their periods will be exactly the same even though one may have an amplitude of 1 mm and the other an amplitude of 1 m. It is assumed that both springs are extended and compressed within their elastic limits.

The function $x = x(t)$ in Equation 4.2(5), gives the position of the particle as a function of the time. The zero point on the time scale (the instant when the stop-watch was set in motion) was chosen when the particle was at the origin and moving in the $+\hat{x}$-direction.

For the chosen solution $x = x(t)$, the velocity of the oscillator is given by

$$v = v(t) = dx/dt = \omega A \cos \omega t = (k/m)^{1/2} A \cos [(k/m)^{1/2} t] \qquad 4.2(6)$$

from which it follows that the maximum speed of the oscillator will be given by $v_{max} = \omega A = (k/m)^{1/2} A$.

The acceleration of the oscillator may be calculated from the velocity.

$$a = a(t) = dv/dt = -\omega^2 A \sin \omega t = -(k/m) A \sin [(k/m)^{1/2} t] \qquad 4.2(7)$$

from which it follows that the maximum magnitude of the acceleration is given by $a_{max} = kA/m$. Newton's second law of motion may now be used to calculate the force as a function of time.

$$F = ma = -m\omega^2 A \sin \omega t = -kA \sin [(k/m)^{1/2} t] \qquad 4.2(8)$$

It is of importance to realise that the expressions for the velocity, acceleration and force which are given in Equations 4.2(6) to 4.2(8), depend on the function $x = x(t)$ which was chosen in 4.2(5). If another solution for x was chosen, the expressions for the other quantities will also be different.

The relationship between the phases of the position, velocity and acceleration, is illustrated in Figure 4.2-2. At time $t = 0$ the position co-ordinate is zero and increasing while the velocity is a maximum and decreasing. At time $t = T/4$, the position co-ordinate has a maximum value and the velocity is zero. From this is can be seen that a **phase difference** exists between the position and the velocity in such a way that the velocity **leads** the position by $\pi/2$. This may also be seen if the expression for the velocity in Equation 4.2(6) is rewritten as a sine function as follows: $v = \omega A \cos \omega t = \omega A \sin(\omega t + \pi/2)$.

A phase difference of of π exists between the position and the acceleration which means that they are always in opposite directions. This is in accordance with the expression for the force in Equation 4.2(1) which led to the differential equation and its solution.

From the graphs it can be seen that the speed is always a maximum when the

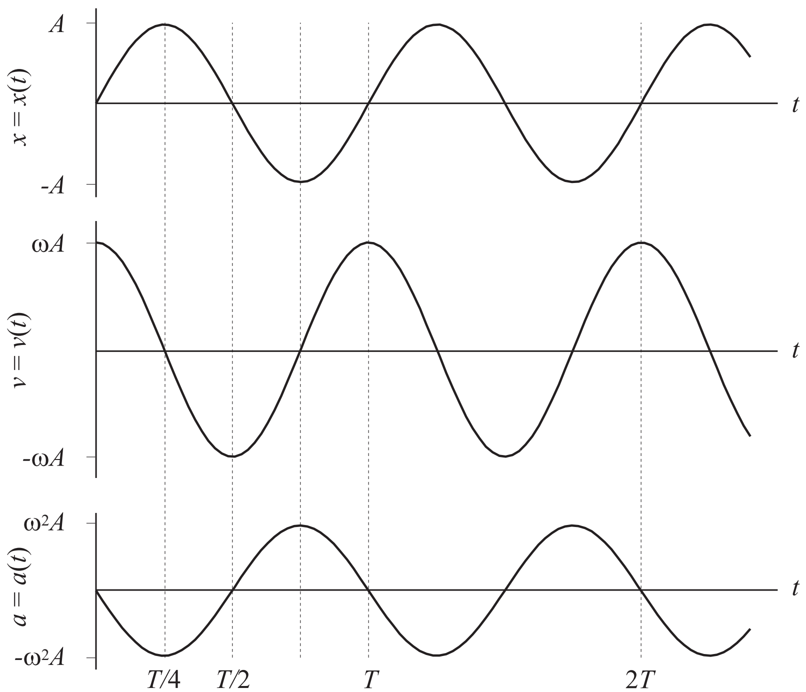


Figure 4.2-2

particle passes through the origin. Then the position and the acceleration are both equal to zero.

The velocity, acceleration and force which are already known as functions of time, may now also be expressed as functions of the position co-ordinate, x. First consider the acceleration. Since $x = A\sin\omega t$, it follows directly from Equation 4.2(7) that

$$a = -\omega^2(A\sin\omega t) = -\omega^2 x = -(k/m)x$$

Various ways exist to express the velocity as a function of the position co-ordinate. From Equation 4.2(6) follows

$$\begin{aligned} v = \omega A\cos\omega t &= \pm\omega[(A\cos\omega t)^2]^{1/2} \\ &= \pm\omega[A^2 - (A\sin\omega t)^2]^{1/2} \\ &= \pm\omega(A^2 - x^2)^{1/2} = \pm[(k/m)(A^2 - x^2)]^{1/2} \qquad 4.2(9) \end{aligned}$$

which is the differential equation that was used at the beginning of the problem.

A more elegant derivation uses the method explained in section 1.6. Use is made of the fact that $v = 0$ when $x = \pm A$. We know that $a\,dx = v\,dv$ and from this follows

$$\begin{aligned} \int_{\pm A}^{x} a\,dx = \int_{\pm A}^{x} -\omega^2 x\,dx &= \int_0^v v\,dv \\ \text{so that} \qquad -\frac{1}{2}\omega^2 x^2|_{\pm A}^{x} &= \tfrac{1}{2}v^2|_0^v \\ \text{from which follows } v = v(x) &= \pm\omega(A^2 - x^2)^{1/2} \\ &= \pm[(k/m)(A^2 - x^2)]^{1/2} \end{aligned}$$

In example 3 of section 2.5.8, the same result was derived by making use of the principle of energy conservation.

4.2.2 The energy of a linear oscillator

In section 2.5 it was pointed out that a conservative force field may be written as the negative gradient of the potential energy function with which it is associated. Thus

$$dE_p/dx = -F_x$$

If this is applied to a linear oscillator, we have

$$dE_p/dx = kx$$

Integration of this differential equation gives the potential energy as a function of the position co-ordinate, x, of the oscillator.

$$E_p = E_p(x) = \frac{1}{2}kx^2 + \text{a constant}$$

As with all problems concerning potential energy, this constant may be chosen to be zero where $x = 0$. With this choice we have

$$E_p = E_p(x) = \frac{1}{2}kx^2 \qquad 4.2(10)$$

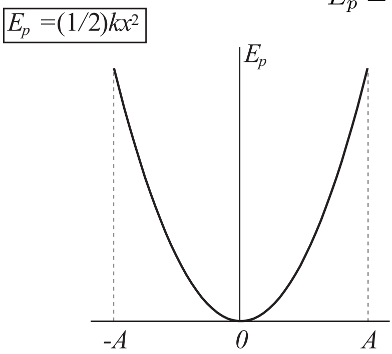


Figure 4.2-3

This result was also calculated in section 2.5.6 when the potential energy of an elastic helical spring was calculated. A graphical representation of this function is shown in Figure 4.2-3. The domain of x is, of course, limited to $-A \leq x \leq A$. As pointed out previously, A is not a property of the system but depends on the initial displacement of the oscillator from its position of stable equilibrium.

By using a suitable solution $x = x(t)$, the potential energy may be written as a function of time. If the solution $x = A\sin\omega t$ is used, we have

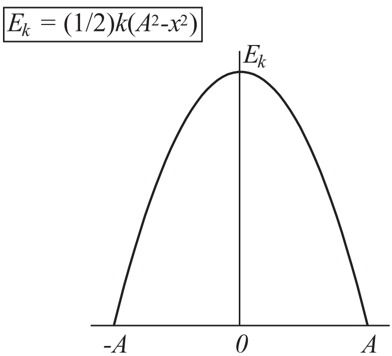


Figure 4.2-4

$$\begin{aligned} E_p &= \frac{1}{2}k(A\sin\omega t)^2 \\ &= \frac{1}{2}kA^2\sin^2\omega t \qquad 4.2(11) \end{aligned}$$

The kinetic energy of the oscillator is given by

$$E_k = \frac{1}{2}mv^2$$

which may be rewritten as either a function of time or a function of the position co-ordinate, x. To express it in terms of x, we use the relationship $v = \pm\omega(A^2 - x^2)^{1/2}$ which is given in Equation 4.2(9).

$$\begin{aligned} E_k &= \frac{1}{2}m\left[\pm\omega(A^2 - x^2)^{1/2}\right]^2 \\ &= \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2), \quad \text{since} \quad \omega^2 = k/m \qquad 4.2(12) \end{aligned}$$

A graphical representation of this function is shown in Figure 4.2-4. The function only exists for x-values in the region $-A \leq x \leq A$. Outside this interval the kinetic energy would be negative which implies an imaginary speed.

If the kinetic energy is to be expressed as a function of time, the expression $v = -\omega A \cos \omega t$ may be used.

$$E_k = \frac{1}{2}m(\omega A \cos \omega t)^2 = \frac{1}{2}kA^2 cos^2 \omega t \qquad 4.2(13)$$

The total energy, E_T, is the sum of the potential and kinetic energy. From Equations 4.2(10) and 4.2(12) it follows that

$$\begin{aligned} E_T = E_p + E_k &= \frac{1}{2}kx^2 + \frac{1}{2}k(A^2 - x^2) \\ &= \frac{1}{2}kA^2 \end{aligned}$$

The same result follows from Equations 4.2(11) and 4.2(13) in which the two forms of energy are expressed as functions of time.

$$\begin{aligned} E_T = E_p + E_k &= \frac{1}{2}kA^2 \sin^2 \omega t + \frac{1}{2}kA^2 \cos^2 \omega t \\ &= \frac{1}{2}kA^2(\sin^2 \omega t + \cos^2 \omega t) \\ &= \frac{1}{2}kA^2 && 4.2(14) \\ &= \frac{1}{2}m\omega^2 A^2 \qquad (\omega^2 = k/m) && 4.2(15) \\ &= 2m\pi^2\nu^2A^2 \qquad (\omega = 2\pi\nu) && 4.2(16) \end{aligned}$$

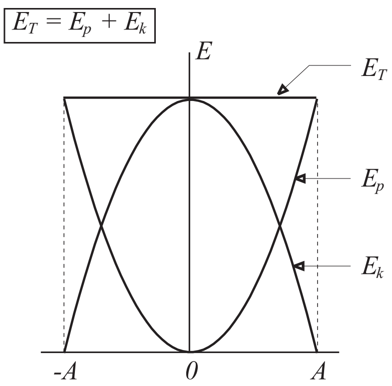


Figure 4.2-5

Equation 4.2(14) shows that the total energy of the oscillator is independent of both the position co-ordinate and time. The graphs in Figure 4.2(5) represent the potential, kinetic and total energy of the oscillator.

An important fact is that the total energy is directly proportional to the *square* of the amplitude for a given frequency.

Since the amplitude is not a property of the system, the total energy will depend on the initial displacement from the position of stable equilibrium.

For a mass suspended from a helical elastic spring, it will depend on the extension (or compression) of the spring from the equilibrium position which is taken as the origin of the frame of reference. The additional energy supplied to the system by the initial displacement from equilibrium, is the total oscillator energy.

The way in which the total energy of the oscillator is expressed in Equation 4.2(16), shows that total energies of two oscillators which have the same mass and amplitude but different frequencies (this implies different force constants), will differ and that they will be in the same ratio as the squares of the frequencies. This means that if the frequency of the one is double that of the other, its energy will be four times that of the latter. If its frequency is triple that of the other, its energy will be nine times as much, etc.

4.2.3 The solution of problems on linear simple harmonic motion

The first fact to establish, is whether the specified system will execute simple harmonic motion if it is disturbed from a position of stable equilibrium. To do this, the expression for the force as a function of the position co-ordinate is used to determine the differential equation. If it is of the form $d^2x/dt^2 = -\omega^2 x$, it will be simple harmonic motion and all the required information may be calculated as it was done in the general treatment of this type of motion.

Pure *free* (in contrast with *damped* or *forced*) simple harmonic motion is a rare phenomenon in nature. The reason is the presence of *dissipative* or other *non-conservative* forces. Dissipative forces are usually a function of the velocity and thus also of position and time. Such forces lead to a slightly more complicated differential equation which will be treated later in this chapter. In the following examples the effects of such forces are disregarded.

In certain cases the differential equation might differ from that for a simple harmonic motion. It will be shown that the correct choice of initial conditions (boundary conditions) will allow approximations which do not differ much from simple harmonic motion.

Examples

1. Consider a light helical spring with force constant k newton per metre. If a mass of m kilograms is suspended from it, the extension of r metres will be in accordance with Hooke's law: $F = mg = kr$ if the elastic limit of the spring is not exceeded. While the mass is at rest, it is in stable equilibrium. When it is given a vertical displacement, it will oscillate about the position of equilibrium.

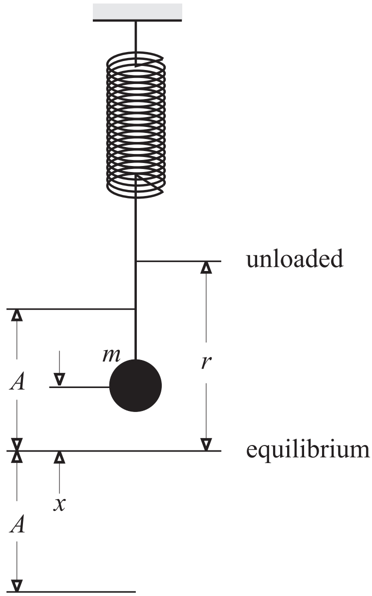


Figure 4.2-6

Choose a frame of reference with its origin at the equilibrium position and $\hat{x}$ vertically upwards. At position x above the equilibrium position, the resultant force on the mass is

$$\bar{F} = -mg\hat{x} + k(r - x)\hat{x} = -kx\hat{x}$$

If the mass of the spring is much less than that of the suspended mass and the maximum magnitude of the drag force (air friction) is much less than maximum magnitude of the restoring force of the spring, their effects may be disregarded and the above equation will describe the motion of the mass fairly well.

The solution of the problem proceeds as illustrated in the general treatment of simple harmonic motion.

In example number 3 of section 2.5.8, the principle of energy conservation was used to derive the following expression:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 + \frac{1}{2}kr^2 = \frac{1}{2}kA^2 + \frac{1}{2}kr^2$$

$$\text{so that} \quad \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

which is the simplest way of obtaining the energy equation of a harmonic oscillator. The quantity $\frac{1}{2}kr^2$ which appears on the left and the right side of the first equation above is a constant quantity which does not play a role in the exchange of energy between the different forms. It is called the **zero-point energy** of the oscillator. This is equal to the elastic energy of the spring when the mass is at rest at the equilibrium position. From the energy equation follows

$$v = \pm[(k/m)(A^2 - x^2)]^{1/2}$$

which represents *two* functions of the velocity in terms of the position coordinate, x, and they apply to the upward (+) and downward (-) motion respectively.

If the positive value is used and v replaced by dx/dt, a first-order separable differential equation which may be solved directly by integration, is obtained.

$$dx/dt = [(k/m)(A^2 - x^2)]^{1/2}$$

$$\begin{aligned}\Rightarrow \qquad (A^2 - x^2)^{-1/2}dx &= (k/m)^{1/2}dt \\ \text{so that} \qquad \arcsin(x/A) &= (k/m)^{1/2}t + \phi \\ \text{and hence} \qquad x &= A\sin\left[(k/m)^{1/2}t + \phi\right]\end{aligned}$$

in which the integration constant, ϕ, may be calculated from the initial conditions. (The integration is done by the substitution $x = A\sin\theta$ from which it follows that $dx = A\cos\theta d\theta$. After the integration has been done the inverse substitution $\theta = \arcsin(x/A)$ is made.)

2.

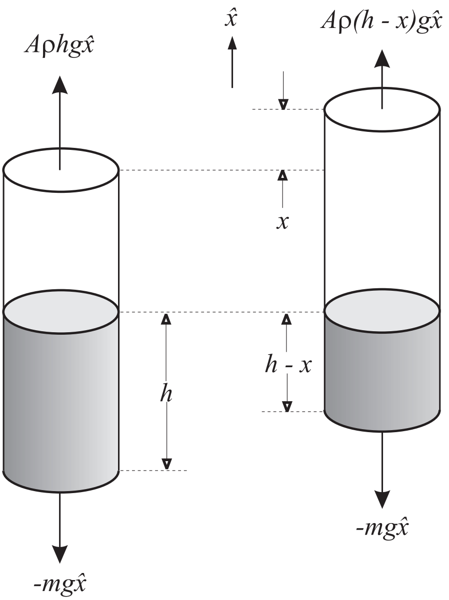


Figure 4.2-7

Consider a cylinder with mass m kg and cross-sectional area A square metres, which floats vertically in a liquid so that a height of h metres is immersed. The density of the liquid is ρ kg m^{-3}.

According to the principle of Archimedes, the buoyant force is equal to the weight of the cylinder so that

$$mg = Ah\rho g$$

Choose a frame of reference with origin on the surface of the liquid and $\hat{x}$ vertically upwards. If the cylinder is displaced vertically, it will oscillate about the equilibrium position. The drag (frictional force between the cylinder and the liquid) will in general be large and it will come to rest in a relatively short time. If the coefficient of viscosity of the liquid (a number which expresses its internal friction) is large enough, no oscillation will occur. If the coefficient of friction and the amplitude of the oscillation are small, the drag may be disregarded as a first-order approximation.

Consider the cylinder when it is at a position x metres above the position of equilibrium. The shaded portions in the sketch represent the volumes of the *displaced* liquid. The resultant force on the cylinder in this position is

$$\bar{F} = -mg\hat{x} + A\rho g(h - x)\hat{x} = -(A\rho g)x\hat{x}$$

This expression represents a restoring force with force constant $A\rho g$ and which will result in simple harmonic motion. The solution and other information are obtained in the same way as for any other simple harmonic motion.

3.

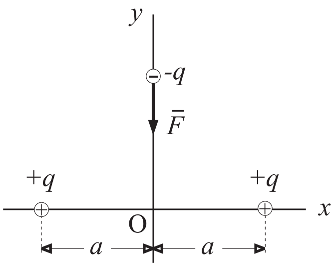


Figure 4.2-8

Consider two fixed equal electric point charges of $+q$ coulomb each at positions $x = +a$ m and $x = -a$ m on the x-axis respectively. A particle of mass m kilogram and electric charge $-q$ coulomb is free to move along the y-axis. Its weight may be disregarded. When the particle is at the origin the resultant force on it is zero and on either side of the origin it experiences a restoring force which tends to move it to the origin. When the particle is at position $\bar{r} = y\hat{y}$ m, this force is given by

$$\bar{F} = -[2kq^2/(a^2 + y^2)^{3/2}]y\hat{y}$$

in which k is the constant which appears in Coulomb's law. The calculation of this force is done in the same way as the identical calculation of the gravitational force on a particle which is shown in example 1 of section 2.2.1.

The force is not linear in y, i.e. not directly proportional to y, and will cause an oscillatory motion which is not simple harmonic motion. If, however, $y \ll a$, the y^2-term in the denominator may be disregarded when compared to a^2 and the resulting expression for the force is then linear in y. For oscillations of which the amplitude is small, the motion will be very nearly simple harmonic with a force constant of $2kq^2/a^3$. Vibrations of atoms in molecules are sometimes treated in this way.

4.3 Angular oscillations

4.3.1 Definitions and properties of simple harmonic angular oscillations.

A system which is constrained to execute rotational motion, may be in a state of stable equilibrium under the action of torques. At the equilibrium position the resultant torque is equal to zero. This means that if the system is given an angular displacement, it will experience a **restoring torque** which will tend to cause motion towards the position of equilibrium. If the magnitude of the torque is directly proportional to the angular displacement, the resultant motion will be a **simple harmonic angular motion**, **simple harmonic rotational motion** or, in short, **rotational SHM**.

$$\tau = -k\theta \qquad \text{or} \qquad \tau + k\theta = 0 \qquad 4.3(1)$$

in which k is called the **stiffness** or **torsion constant**. *Torsion* is synonymous to angular displacement. The restoring torque causes an angular acceleration about the rotational axis which is given by

$$d^2\theta/dt^2 = -(k/I)\theta \qquad \text{or} \qquad d^2\theta/dt^2 + (k/I)\theta = 0 \qquad 4.3(2)$$

in which I is the moment of inertia of the oscillating object about the rotational axis.

At this stage the reader should be able to realise that the mathematics of angular SHM is perfectly analogous to that of linear SHM. The difference is that an angular displacement, θ, is used instead of a linear position co-ordinate such as x. The first and second derivatives of the angle are the angular velocity, $\dot{\theta} = d\theta/dt$, and the angular acceleration, $\ddot{\theta} = d^2\theta/dt^2$, respectively. The symbol ω which was used for angular velocity in chapter 3, cannot be used for that purpose in this chapter since it was chosen to indicate angular frequency. For this reason the symbol $\dot{\theta}$ is used.

The most important results for an angular SHM are given below. The reader is advised to work through their deductions from the definitions in the same way as was done for linear SHM.

$$\omega = 2\pi\nu = 2\pi/T = (k/I)^{1/2} \qquad 4.3(3)$$

$$\text{so that} \quad T = 2\pi(I/k)^{1/2} \quad \text{and} \quad \nu = (1/2\pi)(k/I)^{1/2} \qquad 4.3(4)$$

For the initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) > 0$, the solution for the differential equation is as follows for an amplitude of θ_0:

$$\theta = \theta_0 \sin \omega t = \theta_0 \sin\left[(k/I)^{1/2} t\right] \qquad 4.3(5)$$

The angular velocity of the oscillator is

$$\dot{\theta} = \omega\theta_0 \cos \omega t = (k/I)^{1/2}\theta_0 \cos\left[(k/I)^{1/2} t\right] \qquad 4.3(6)$$

and its angular acceleration

$$\ddot{\theta} = -\omega^2\theta_0 \sin \omega t = -(k/I)\theta_0 \sin\left[(k/I)^{1/2} t\right] \qquad 4.3(7)$$

The relationships between the phases of these quantities are the same as that shown by the graphs in Figure 4.2-2 for linear SHM.

By the same methods used for linear oscillations, the angular velocity may be calculated as a function of the angular displacement.

$$\dot{\theta} = \pm\omega(\theta_0^2 - \theta^2)^{1/2} = \pm[(k/I)(\theta_0^2 - \theta^2)]^{1/2} \qquad 4.3(8)$$

The angular potential energy is given by

$$E_p = \frac{1}{2}k\theta^2 = \frac{1}{2}I\omega^2\theta_0^2 \sin^2 \omega t \qquad 4.3(9)$$

and the rotational kinetic energy by

$$E_k = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}I\omega^2\theta_0^2\cos^2\omega t \qquad 4.3(10)$$

The total energy of the oscillator is the sum of the potential and kinetic energy.

$$E_T = \frac{1}{2}k\theta_0^2 \qquad 4.3(11)$$

$$= \frac{1}{2}I\omega^2\theta_0^2 \qquad 4.3(12)$$

$$= 2I\pi^2\nu^2\theta_0^2 \qquad 4.3(13)$$

4.3.2 The torsion oscillator

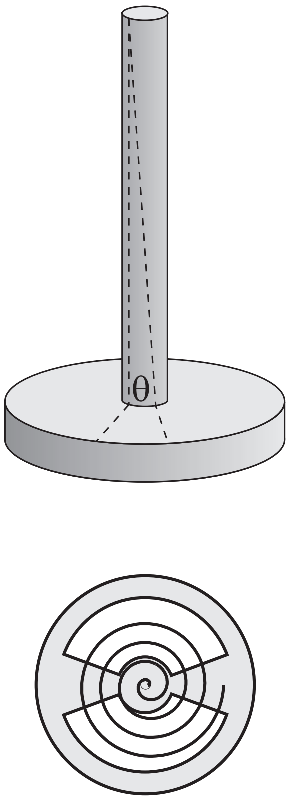


Figure 4.3-1

Consider a solid vertical rod consisting of an elastic material and which is tightly clamped at its upper end. The free end may be twisted to cause a torsion of θ with a resulting restoring torque. Within the elastic limit, the torque will be directly proportional to the torsion as shown in Equation 4.3(1). If a body is fixed to the free end, it will execute angular SHM if friction is not present or may be disregarded. Such an oscillator is called a **torsion oscillator**. Before the differential equation can be set up, it is necessary to know the torsion constant of the rod and the moment of inertia about the axis of rotation.

A similar set-up which will, but for minor frictional drag, execute angular SHM, is the **balance wheel** of a mechanical wrist watch or alarm clock. The restoring torque is supplied by a small spiral spring which is deformed to both sides of the equilibrium position. The balance wheel is attached to an escapement through which it regulates the average angular velocity of the clock hands. See Figure 4.3-1.

4.3.3 The simple pendulum

The **simple pendulum** is an idealised model of a real pendulum in which the oscillating mass, which is called a **pendulum bob** is taken to be a point mass while the mass of the suspension is disregarded. Some prefer to call this ideal model a **mathematical pendulum**. A small pendulum bob of high density which is suspended from a very light cord will be a good approximation.

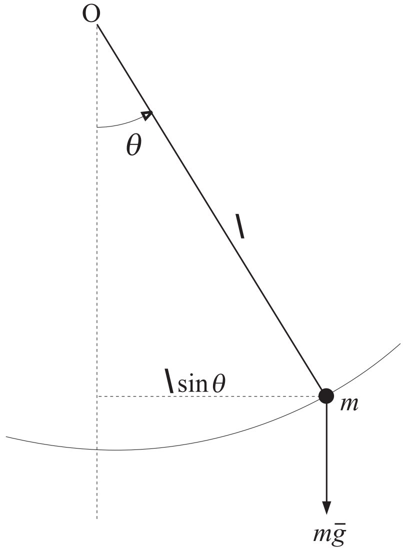


Figure 4.3-2

Let the mass of a simple oscillator (shown in Figure 4.3-2) be m kilograms and the length of the suspension, ℓ metres. When the mass is at rest directly under the suspension point, it experiences stable equilibrium and will stay there until disturbed. When it is given an angular displacement, it will oscillate about the equilibrium position.

Consider an oscillating simple pendulum when the angular displacement is $+\theta$ radians. The torque of its weight about the suspension point is

$$\tau = -mg \times \ell \sin\theta$$

The minus sign is necessary because the torque is always opposite to the angular displacement, i.e. the torque is a *restoring* torque. The torque causes an angular acceleration, $\alpha = d^2\theta/dt^2$. According to Newton's second law we have

$$\tau = I\alpha = I\, d^2\theta/dt^2$$

From this follows the differential equation for the motion.

$$d^2\theta/dt^2 = -(mg\ell/I)\sin\theta$$

The right-hand side of the equation is not linear in θ and although it describes an angular oscillation, it is not a simple harmonic oscillation. The exact solution of this equation falls outside the scope of this book. If, however, the amplitude is small, the sine of the angular displacement may be approximated by the angle expressed in radians. If this is done the differential equation becomes

$$d^2\theta/dt^2 \approx -(g/\ell\,)\theta$$

The smaller the angular displacement, the better the approximation which is a good description of angular SHM. The angular frequency is given by

$$\omega = (g/\ell)^{1/2} = 2\pi\nu = 2\pi/T$$

and the period of the pendulum is

$$T = 2\pi(\ell/g)^{1/2} \qquad 4.3(14)$$

4.3.4 The physical or compound pendulum

If the mass distribution of a pendulum is such that it cannot be approximated by a point mass, it is known as a **physical** or **compound pendulum**.

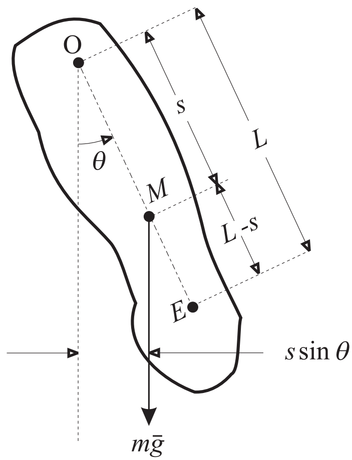


Figure 4.3-3

Consider a body with mass m kilograms which may oscillate about an axis O (shown in Figure 4.3-3). The centre of mass is indicated by M and $OM = s$ metres. The moment of inertia of the body about a horizontal axis through O is indicated by I_O. The differential equation may be set up in exactly the same manner as for a simple pendulum.

$$\begin{aligned} d^2\theta/dt^2 &= -(mgs/I_O)\sin\theta \\ &\approx -(mgs/I_O)\theta \qquad (\theta \text{ small}) \end{aligned}$$

The period of the pendulum is given by

$$T = 2\pi(I_O/mgs)^{1/2} \qquad 4.3(15)$$

In Equation 4.3(15), the moment of inertia, I_O, depends not only on the mass and shape of the body which oscillates, but also on the position where it is suspended. One and the same body may thus have an infinite number of periods depending on where the pivot is.

If L is the length of the simple pendulum which has the same period as a given physical pendulum, L is called its **equivalent simple pendulum length**. From Equations 4.3(14) and 4.3(15) follows

$$\begin{aligned} T &= 2\pi(L/g)^{1/2} = 2\pi(I_O/mgs)^{1/2} \\ \text{so that} \qquad L &= I_O/ms \qquad 4.3(16) \end{aligned}$$

Consider point E (see Figure 4.3-3) which lies on the extension of line OM so that $OE = L$. This position is known as the **centre of oscillation** or the **centre of percussion**. It should be kept in mind that this position does not necessarily lie within the body as shown in the sketch. If it does lie within the body, it represents the position of a point mass which oscillates at its **natural frequency**, i.e. that point mass which behaves as a simple pendulum with length L. Points closer to O are forced to oscillate at a slower rate than their natural frequencies and those further away than E at a higher frequencies.

An interesting property of the centre of percussion becomes clear when the period of the oscillation is calculated when the body is pivoted there. For the calculation of this period, T', Equation 4.3(15) is used but the moment of inertia is then I_E and the distance between the pivot and the centre of mass, $L - s$. (See Figure 4.3-3). From Equation 4.3(15) follows

$$T' = 2\pi(I_E/mg[L - s])^{1/2} \qquad 4.3(17)$$

If I_M is the moment of inertia of the body about an axis through its centre of mass, it follows from Steiner's theorem for parallel axes that

$$I_E = I_M + m(L - s)^2 \qquad \text{and} \qquad I_O = I_M + ms^2$$

Substitute the above expression for I_O in Equation 4.3(16).

$$\begin{aligned} L = (I_M + ms^2)/ms &= I_M/ms + s \\ \text{so that} \quad L - s &= I_M/ms \end{aligned} \qquad 4.3(18)$$

Also substitute the expression for I_E in Equation 4.3(17).

$$T' = 2\pi[I_M/mg\{L - s\}) + (L - s)/g]^{1/2}$$

which simplifies as follows if the expression for $L - s$ in Equation 4.3(18) is used:

$$\begin{aligned} T' &= 2\pi(s/g + I_M/mgs)^{1/2} \\ &= 2\pi([I_M + ms^2]/mgs)^{1/2} \\ &= 2\pi(I_O/mgs)^{1/2} = T \end{aligned}$$

The body thus oscillates about E at the same frequency as when oscillating about O. Any two points which comply with this condition, are known as **conjugate points**.

In order to derive a more general result, the existence of pairs of conjugate points may also be shown in a different manner. From the differential equation for a physical pendulum it follows that

$$\begin{aligned} \omega^2 &= mgs/I_O \\ &= mgs/(I_M + ms^2) \qquad \text{(Steiner's law)} \end{aligned}$$

If $\Re$ is the radius of gyration of the body about an axis through its centre of mass which is parallel to that through O, then $I_M = m\Re^2$. Substituting this in the above equation, gives the following quadratic equation for s:

$$s^2 - (g/\omega^2)s + \Re^2 = 0 \tag{4.3(19)}$$

which is independent of the mass of the body.

From the theory of quadratic equations it is known that two real roots, s_1 and s_2 exist for each admissible value of ω in accordance with the relationship $g^2/\omega^4 \geq 4\Re^2$. This relationship shows that a maximum value of the angular frequency, ω, exists and accordingly a minimum value of the period, $T = 2\pi/\omega$. The minimum value of the period corresponds to the condition that $g^2/\omega^4 = 4\Re^2$ and is given by

$$T_{min} = 2\pi(2\Re/g)^{1/2}$$

The period is thus a minimum when $s_1 = s_2 = \Re$.

From the theory of quadratic equations, it follows that the sum of the two roots of Equation 4.3(19) is given by

$$\begin{aligned} s_1 + s_2 &= g/\omega^2 \\ &= g/(mgs/I_O) = I_O/ms \end{aligned}$$

which is equal to L, the length of the equivalent mathematical pendulum.

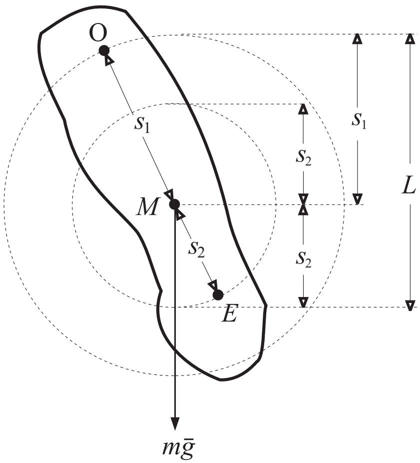


Figure 4.3-4

In Figure 4.3-4 the same sketch is shown as in Figure 4.3-3 and the new symbols are indicated on it. From this sketch it can be seen that all the possible positions for the pivot which will give one and the same frequency, lie on two concentric circles with radii s_1 and s_2. The centre of mass is the common centre of the circles and the said positions do not necessarily lie in the body.

The product of the two roots of the equation is given by

$$s_1 \times s_2 = \Re^2 \tag{4.3(20)}$$

The last interesting fact of the centre of oscillation relates to the term *centre of percussion*. In order to investigate this property, consider a compound pendulum which consists of a rigid rod sus-

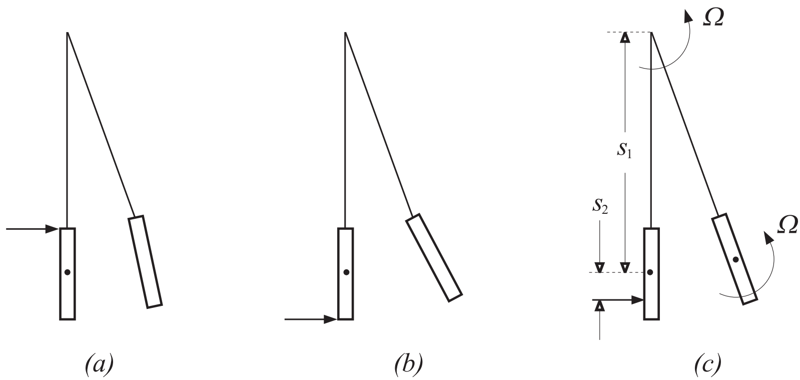


Figure 4.3-5

pended from a light cord as shown in Figure 4.3-5. If the pendulum is at rest and a sharp horizontal blow is applied to the rod, in general an angle will develop between the rod and the cord. If the position where the blow is struck is relatively high on the rod, the angle will be as shown in sketch (a) and if it is low, as in sketch (b). Between these two positions is one where no angle will develop and the condition for which this will be the case, is that the rod will have the same angular velocity, Ω, about its centre of mass as that which the centre of mass will have about the pivot.

Suppose the centre of mass is s_1 metres below the pivot and the position below the centre of mass where the rod is struck without causing the development of an angle, equal to s_2. (See sketch (c).) If the rod receives an impulse of magnitude Δp, the ensuing initial angular momentum about its centre of mass will be

$$J = s_2 \Delta p = m\Re^2 \Omega \qquad \ldots (1)$$

If the centre of mass has a horizontal velocity of $\bar{v}$ just after the blow, the impulse is given by

$$\Delta p = mv = ms_1 \Omega \qquad \ldots (2)$$

Dividing Equation (1) by Equation (2) gives $s_2 = \Re^2/s_1$ which may be rewritten as

$$s_1 s_2 = \Re^2$$

This is the same result as that in Equation 4.3(20) which shows that the impact point which complies with the condition that no angle will tend to develop in the pendulum, is the centre of oscillation or percussion.

A practical implication of this result is found in the use of a cricket bat, an axe or a hammer, etc. The arm of the user and the instrument which is used, form a pendulum. If the point of impact is not the centre of percussion, the user will feel a sting in his hands with each blow.

Example:

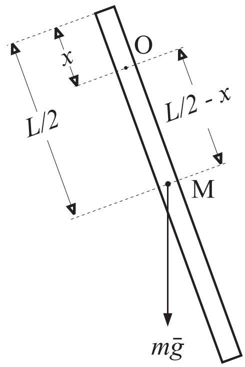


Figure 4.3-6

A uniform homogeneous thin rod is 2 metres long and may be pivoted at any position along its entire length is such a way that is will oscillate in Earth's gravitational field. The distance from the pivot to the one end is indicated by x. Calculate the period of the pendulum as a function of x in the interval $0 \leq x \leq 2$ metres.

The period of a compound pendulum is given by Equation 4.3(15).

$$T = 2\pi(I_O/mgs)^{1/2}$$

in which I_O is the moment of inertia about the pivot. It follows from Steiner's law for parallel axes that

$$I_O = \frac{1}{12}mL^2 + m(L/2 - x)^2$$

in which m is the mass of the rod and $L = 2$ metres, its length. If this expression for I_O and $s = (L/2) - x$ is substituted in the period, we obtain the required function

$$T = \frac{2\pi}{\sqrt{3g}}\left(\frac{4 - 6x + 3x^2}{1 - x}\right)^{1/2}$$

in which the value $L = 2$ metre is used. If $g = 10\ \mathrm{m\,s^{-2}}$ is used the graph shown in Figure 4.3-7 is obtained.

From the graph a number of interesting properties, some of which became apparent during the study of physical pendulums, can be seen. Firstly it can be seen that the period has a certain minimum which corresponds to two different values of x. These two positions are symmetrical about the centre of mass of the rod. When x approaches half of the length of the rod, the period tends to infinity. This is simple to understand since a body suspended at its centre of mass, cannot experience a restoring torque due to gravity. Suspended at the centre of mass, the rod experiences neutral equilibrium. Within limits, four values of x exist which all give the same period. These positions are conjugates

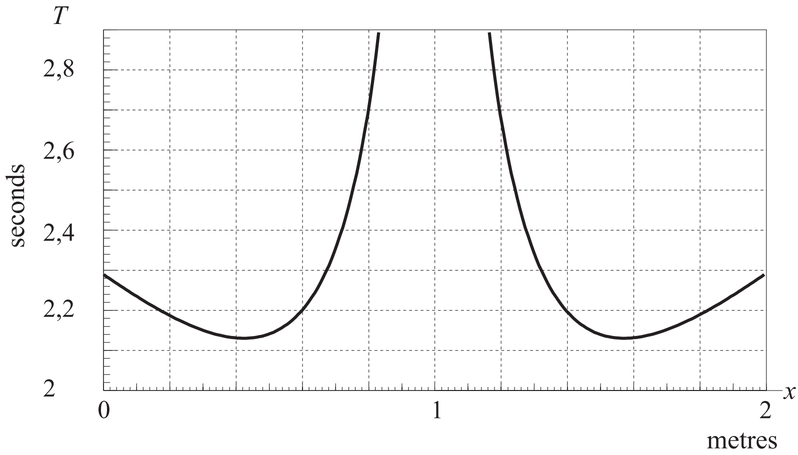


Figure 4.3-7

which lie on the circumferences of two circles as shown in Figure 4.3-4.

4.4 Damped oscillations

The presence of dissipative forces were disregarded in all the problems concerning oscillators up to now. In practice most oscillators are subject to **damping** which is caused by the presence of frictional forces. The work done against such forces irreversibly removes energy of an oscillator in the form of heat. The rate at which energy is lost by an oscillator, depends on the nature of the dissipative force or forces which are present. A simple case is that where the frictional force is directly proportional to the velocity of the oscillator. Such cases occur when an oscillator is damped by a **viscous fluid** (liquid or gas). In such cases the frictional force is given by

$$F_f = -cv = -c(dx/dt)$$

in which c is a constant. The negative sign indicates that the frictional force is always in the opposite direction of that of the velocity. The differential equation for such a damped motion is as follows:

$$m(d^2x/dt^2) + c(dx/dt) + kx = 0 \qquad 4.4(1)$$

which is usually written in the following form for reasons which will presently become apparent:

$$d^2x/dt^2 + 2\gamma(dx/dt) + \omega_0^2 x = 0$$

in which $2\gamma = c/m$ and $\omega_0 = (k/m)^{1/2}$ = the natural frequency which the system would have if it was not damped. The nature of the solution of the equation will depend on the relative magnitudes of γ and ω_0.

For a case in which γ is less than ω_0, the system will oscillate and will be so-called **lightly damped** or **underdamped**. If $\gamma \ll \omega_0$ such an oscillation should have a very good resemblance to one which is not damped but one would expect that it will eventually come to rest at the equilibrium position. By substitution in the differential equation, the reader may verify that the following function is a solution:

$$x = A_0 e^{-\gamma t} \sin(\omega t + \phi) \qquad 4.4(2)$$

in which $$\omega = (\omega_0^2 - \gamma^2)^{1/2} \qquad 4.4(3)$$

The constants A_0 and ϕ may be calculated from the initial conditions.

From this solution two important characteristics of a lightly damped oscillator may be seen. The first property is that it does not oscillate at its natural undamped frequency but at a lower frequency given by

$$\nu = \omega/2\pi = (1/2\pi)(\omega_0^2 - \gamma^2)^{1/2}$$

If $\omega_0 \gg \gamma$ the difference between this frequency and that for an undamped oscillation will be small and often negligible over a limited interval of time.

The second interesting property is that the amplitude is a function of time.

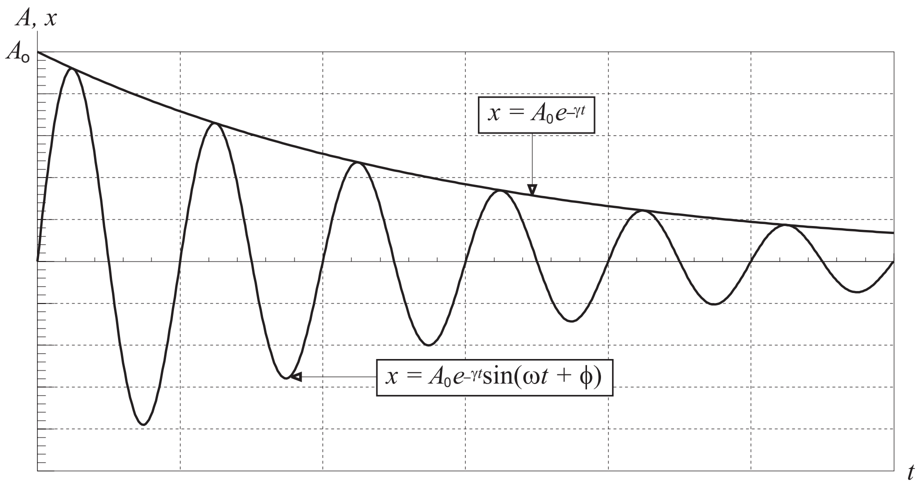


Figure 4.4-1

$$A = A_0 e^{-\gamma t}$$

The quantity $1/\gamma$ is known as the **time constant** of the exponential decay of the amplitude during which the amplitude will diminish to $1/e$ of its initial value. Some prefer to use the **half-life interval** (**halving interval**), $\tau = t_{1/2}$ which is the interval of time during which the amplitude will diminish by half. The reader can verify that this is given by

$$\tau = t_{1/2} = (\ln 2)/\gamma$$

Figure 4.4-1 shows a possible graph for Equation 4.4(2)

In cases where $\gamma > \omega_0$ oscillation will not occur and the motion is described as **overdamped**. The solution of the differential equation for overdamping is

$$x = A_0 e^{-(\gamma+\lambda)t} + B_0 e^{-(\gamma-\lambda)t} \qquad 4.4(4)$$

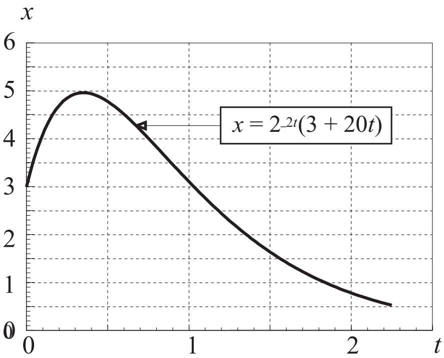


Figure 4.4-2

in which $\lambda = (\gamma^2 - \omega_0^2)^{1/2}$ and the constants A_0 and B_0 depend in the initial conditions. The solution is not oscillatory. After an initial disturbance the system will return exponentially to its state of stable equilibrium.

When $\gamma = \omega_0$ we have so-called **critical damping**. The solution for critical damping is

$$x = e^{-\gamma t}(C_0 + D_0 t) \qquad 4.4(5)$$

in which the constants C_0 and D_0 follow from the initial conditions. A critically damped system is also non-oscillatory and returns exponentially to stable equilibrium after an initial disturbance. Figure 4.4-2 shows the graph for a critically damped system. The basic difference between a critically damped system and an overdamped system is that the former reaches its equilibrium position in a much shorter interval of time.

Without shock-absorbers or with shock-absorbers which do not function as they should, a vehicle equipped with elastic springs will oscillate after each vertical disturbance. A trip in such a vehicle is not only unpleasant but may also be dangerous. Since a critically damped system will restore in the shortest time interval, the ideal is that shock-absorbers on a vehicle should be such that the system is critically damped. Since the load on a vehicle is not constant, it would probably be correct to state that all vehicles equipped with elastic springs and shock-absorbers, are slightly overdamped.

4.5 The superposition of linear SHM's

If a system is forced to simultaneously execute two or more different simple harmonic oscillations, the action is described as the **superposition** or combination of oscillations. The result is known as **interference**. Interference phenomena are interesting only under some special circumstances. In this discussion a number of such cases will be treated.

4.5.1 Two parallel oscillations with equal frequencies

Consider the superposition of the following two harmonic oscillations:

$$x_1 = A \sin(\omega t + \alpha) \qquad \text{and} \qquad x_2 = B \sin(\omega t + \beta)$$

The resultant oscillation is given by the sum of these two functions.

$$x = A \sin(\omega t + \alpha) + B \sin(\omega t + \beta)$$

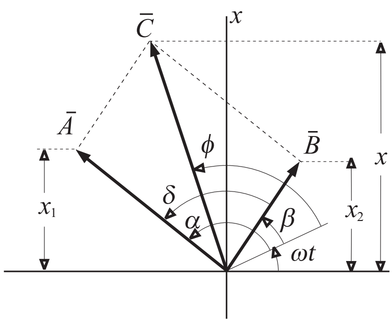


Figure 4.5-1

The phasor diagram which represents this addition, is shown in Figure 4.5-1. Oscillation x_1 is represented by phasor $\bar{A}$ which rotates counter-clockwise at angular velocity ω from the initial position α, and oscillation x_2 by phasor $\bar{B}$ which rotates at the same angular velocity from the initial position β. The sketch shows the conditions which exist after t seconds have elapsed. The resultant oscillation may be determined by the addition of the projections of $\bar{A}$ and $\bar{B}$ on the x-axis while taking their signs into account.

A more elegant way is to determine the vector sum of $\bar{A}$ and $\bar{B}$ to give phasor $\bar{C}$. The projection of phasor $\bar{C}$ on the x-axis gives the numerical value of the resultant oscillation at the given instant. The equivalence of these two approaches forms the basis of vector algebra.

It may be seen from the sketch that the resultant oscillation is given by the function

$$x = C \sin(\omega t + \phi) \qquad 4.5(1)$$

in which C is given by the expression

$$C = (A^2 + B^2 + 2AB \cos\delta)^{1/2} \quad \text{in which} \quad \delta = |\alpha - \beta| \qquad 4.5(2)$$

$$\text{and} \qquad \phi = \arctan\left(\frac{A\sin\alpha + B\sin\beta}{A\cos\alpha + B\cos\beta}\right) \qquad 4.5(3)$$

The phasor $\bar{C}$ rotates at the same angular velocity as the other two, and the angles between them remain unaltered as time progresses. It is said that the phasor diagram is in a **steady state** and it means that its shape does not change although it rotates.

The same result may be obtained if the functions for the oscillations which are to be compounded, are first expanded (use the formula for the sine of compound angles) and then added. This is left as an exercise to the reader.

The resultant oscillation is thus also a simple harmonic oscillation with the same frequency as that of the two which were superposed. The phase constant of the resultant oscillation will, in general, differ from that of x_1 and x_2. The amplitude, C, will also depend on the **phase difference**, $\delta = |\alpha - \beta|$ between x_1 and x_2.

For the special case when $\alpha = \beta$, their numerical value is equal to ϕ so that phasors $\bar{A}$, $\bar{B}$ and $\bar{C}$ coincide. It is said that the oscillations are **in phase**. For this special case the resultant amplitude has a maximum, namely, $C = A + B$. As in this case, **constructive interference** will always occur when the phase difference is equal to $n.2\pi$ in which n is an integer. Such an example is shown in Figure 4.5-2(a).

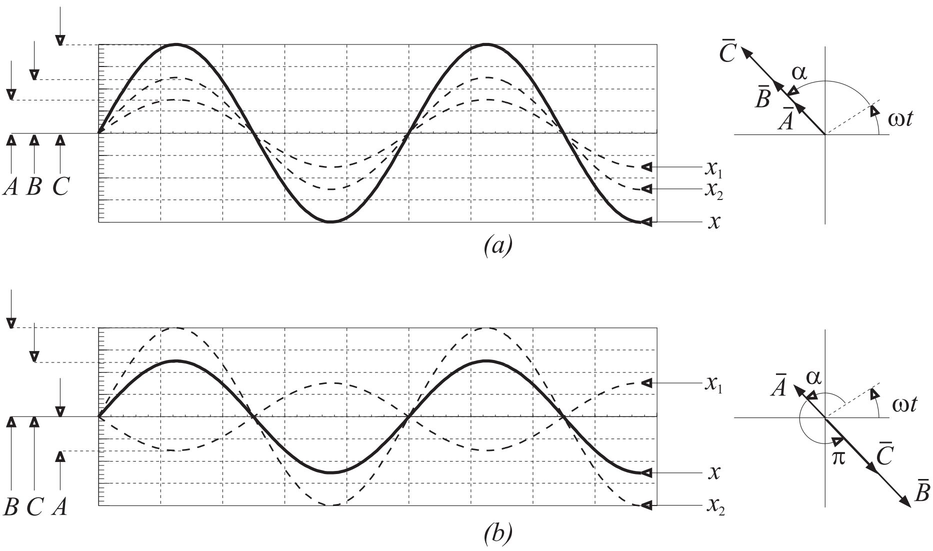


Figure 4.5-2

When $|\alpha - \beta| = \pi$, the opposite occurs. $\bar{A}$ and $\bar{B}$ are anti-parallel and the amplitude of the resultant oscillation has its smallest possible value namely $C = |A - B|$. The phasor $\bar{C}$ will be parallel to the phasor $\bar{A}$ or $\bar{B}$ of which the magnitude is the greatest. This is known as **destructive interference** which occurs at a phase difference of $(2n+1)\pi$ in which n is an integer. Such a case is illustrated by Figure 4.5-2(b).

4.5.2 Two parallel oscillations with different frequencies

In this section the superposition of two harmonic oscillations with different frequencies will be studied. It will presently be clear that no loss of generality will occur when it is chosen that $\alpha = \beta = 0$. Consider the following two oscillations:

$$x_1 = A \sin \omega_1 t \qquad \text{and} \qquad x_2 = B \sin \omega_2 t$$

The resultant oscillation is given by the sum of these two functions.

$$x = A \sin \omega_1 t + B \sin \omega_2 t$$

In section 4.5.1 where the frequencies of the two oscillations were identical, the phasor diagram was steady. This allowed us to reduce the expression for the resulting oscillation to a simple form of which the behaviour is quite obvious. In the present case, however, the frequencies are different and the phasors $\bar{A}$ and $\bar{B}$ rotate at different angular velocities so that the phasor diagram constantly changes from one instant to the next. The parallelogram which is formed when the hands of an analogue clock are used as two adjacent sides of a parallelogram, is a perfect example of the phasor diagram under consideration. The only useful information which can be gleaned from this phasor diagram is the amplitude of the resultant oscillation. As before (see Figure 4.5-1 and keep in mind that the phase constants are equal to zero), we have at any instant t that

$$C = (A^2 + B^2 + 2AB\cos\delta)^{1/2} \qquad \text{with} \qquad \delta = |\omega_1 t - \omega_2 t| \qquad 4.5(4)$$

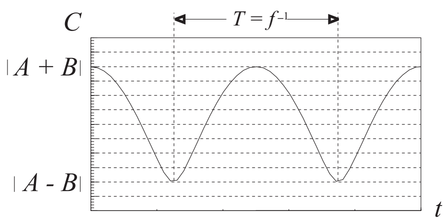


Figure 4.5-3

Since δ increases with time, the magnitude of C will oscillate between two extreme values as $\cos\delta$ assumes values which lie between -1 and $+1$. The maximum values are given by $|A + B|$ and they occur when $\cos\delta = 1$, i.e. when $\delta = n.2\pi$. The minimum values are $|A - B|$ and they occur when $\delta = (2n+1)\pi$. In both cases n is an integer.

The amplitude, C, is an oscillatory function in itself. An amplitude which varies with time, is called a **modulated amplitude**. The frequency, f, with which the amplitude changes is given by

$$\begin{aligned} 2\pi f &= |\omega_1 - \omega_2| \\ \Rightarrow\ f &= |\omega_1 - \omega_2|/2\pi \\ &= |\nu_1 - \nu_2| \end{aligned} \qquad 4.5(5)$$

in which the frequencies of the oscillations which are superimposed, are given by $\nu_1 = \omega_1/2\pi$ and $\nu_2 = \omega_2/2\pi$. Figure 4.5-3 shows the variation of the amplitude with time. It is a graphical representation of the function in Equation 4.5(4).

This regular change in the amplitude is known as **beats** and may usually be observed only when the frequencies which are superimposed are relatively large when compared to their difference. When the difference is large, beats as such can usually not be discerned.

When a vibrating tuning fork is held near the ear of an observer, the ear-drum executes SHM and this is perceived as a musical sound. When a second tuning fork with a slightly different frequency is sounded simultaneously with the first, the observer hears a musical sound with beats of which the frequency is equal to the difference of the two primary frequencies. It will be shown later that the frequency of the new sound is the average of that of the two tuning forks.

Beats are useful when two oscillators have to be tuned to the same frequency. The person who does the tuning activates both oscillators and changes the frequency of one until beats cannot be heard. This principle is applied by piano tuners. The pilot of a propeller-driven aircraft with twin motors uses beats to adjust both engines to the same angular velocity. The principle is also used when two electronic oscillators have to be adjusted to have equal frequencies.

If the amplitudes of the two primary oscillations are equal, it is quite simple to derive an expression for the resultant oscillation from which its properties follow clearly. In this derivation the following goniometric identity is required:

$$\sin P + \sin Q = 2\cos\frac{1}{2}(P-Q)\sin\frac{1}{2}(P+Q)$$

Consider the superposition of the following two harmonic oscillations:

$$x_1 = A\sin\omega_1 t \qquad \text{and} \qquad x_2 = A\sin\omega_2 t \qquad \text{with} \qquad \omega_1 > \omega_2$$

The resultant oscillation is given by

$$\begin{aligned} x &= A\sin\omega_1 t + A\sin\omega_2 t \\ &= 2A\cos\frac{1}{2}(\omega_1 - \omega_2)t\sin\frac{1}{2}(\omega_1 + \omega_2)t \end{aligned} \qquad 4.5(6)$$

This result may be written in the form

$$\begin{aligned} x &= C \sin \omega t \qquad \text{in which} \\ \omega &= \frac{1}{2}(\omega_1 + \omega_2) \quad \text{and} \quad C = C(t) = 2A \cos \frac{1}{2}(\omega_1 - \omega_2)t \end{aligned}$$

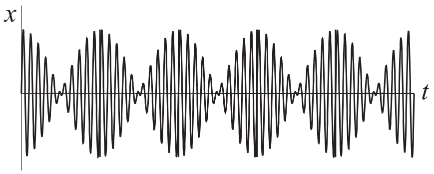


Figure 4.5-4

From this it should be clear that the function represents a simple harmonic oscillation of which the amplitude varies with time. The frequency at which the amplitude changes, is equal to the difference between the frequencies of the two oscillations which are superimposed. Because the two initial frequencies are equal the resultant amplitude varies between zero and $2A$. The frequency of the resultant oscillation is equal to the average of the two primary frequencies. Figure 4.5-4 shows $x = x(t)$ which is given by Equation 4.5(6).

4.5.3 Two perpendicular oscillations

Consider the following pair of perpendicular oscillations:

$$x = A \sin \omega_x t \qquad \text{and} \qquad y = B \sin (\omega_y t + \phi)$$

It might seem strange that a phase constant has been chosen for the oscillation parallel to $\hat{y}$ and not for the one parallel to $\hat{x}$. This corresponds to the choice that $x(0) = 0$ with $dx/dt > 0$. Any other choice would have given the same result but an extra phase constant would incur extra calculations in describing the system without any additional physical difference.

The resultant oscillation is given by

$$\bar{r} = (A \sin \omega_x t)\hat{x} + (B \sin [\omega_y t + \phi])\hat{y}$$

The oscillating particle moves on a plane and the parameter t can be eliminated only in a few very simple cases to give the space curve in Cartesian co-ordinates which will allow the reader to recognise it. Since it does not have much use, these possibilities will not be pursued.

The shapes of the space curve are interesting and sometimes quite beautiful when the ratio between the two frequencies can be described by two integer numbers. The shapes of the space curves which develop in such cases, are known as **Lissajous figures**. These figures afford a useful and very sensitive method to compare the unknown frequency of an oscillator to that of one which is known.

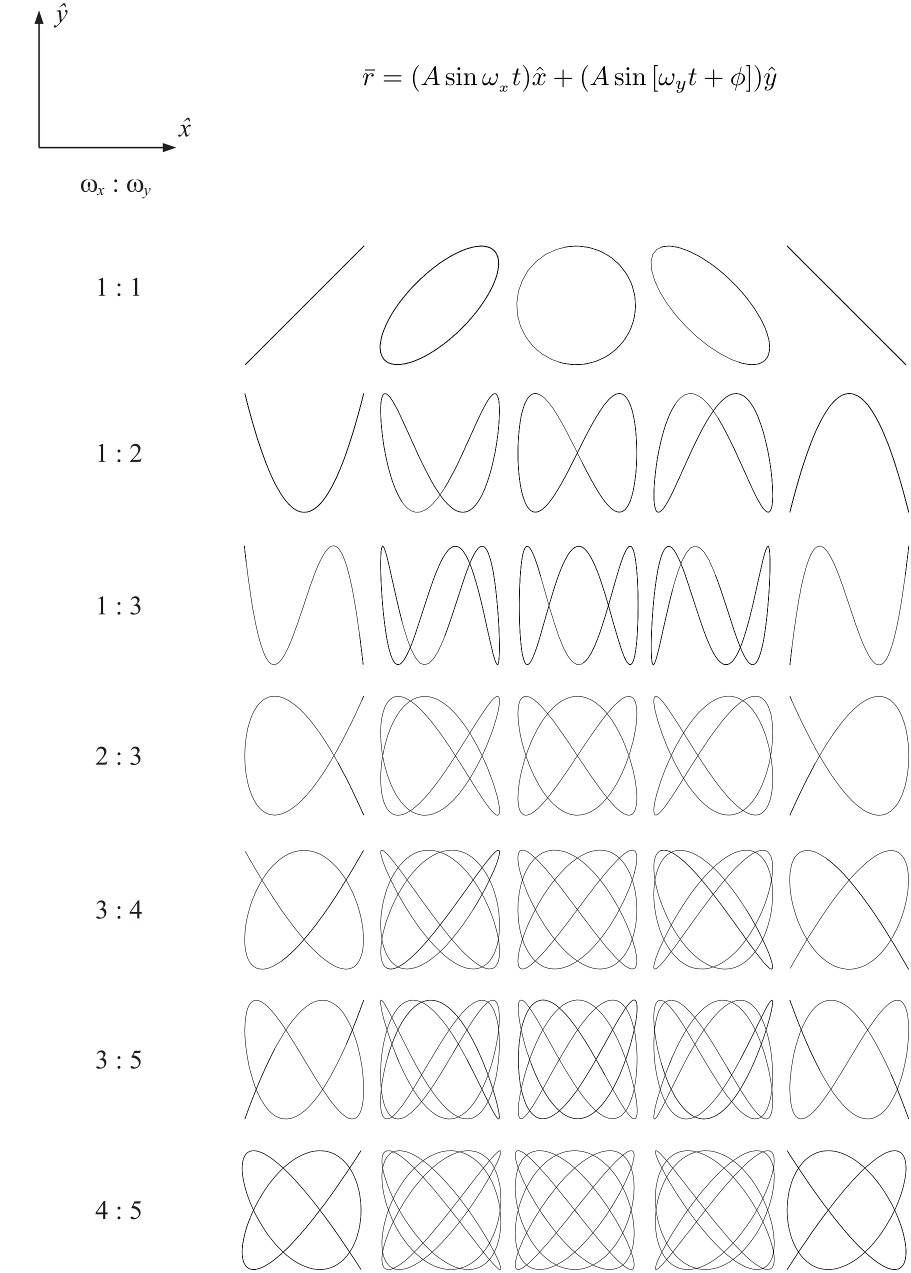

$\hat{y}$
$\hat{x}$
$\bar{r} = (A \sin \omega_x t)\hat{x} + (A \sin [\omega_y t + \phi])\hat{y}$
$\omega_x : \omega_y$
1 : 1
1 : 2
1 : 3
2 : 3
3 : 4
3 : 5
4 : 5

Figure 4.5-5

The reader should be able to determine the ratio of the two relevant frequencies from a given Lissajous figure. This is done by starting at any point on the closed figure (all Lissajous figures are closed) and whilst tracing the figure, the left-right motions are counted. The process is repeated and the up-down motions counted. The ratio between the two answers is the ratio between the two frequencies.

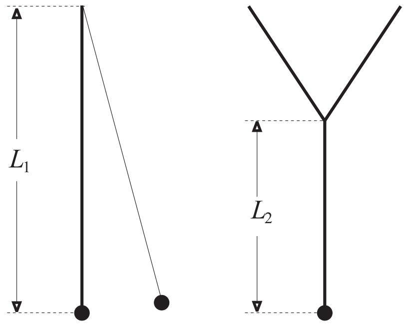


Figure 4.5-6

The simplest way to make Lissajous figures visible is by using an oscilloscope and two electronic signal generators to drive the beam along the x and y-directions respectively. A pendulum of which the suspension cord is Y-shaped as shown in Figure 4.5-6 will have one frequency when swung in the plane of the Y and another frequency when it is swung in a direction perpendicular to this plane. The pendulum bob describes Lissajous figures when the square root of the ratio between the two pendulum lengths is a fraction of which the numerator and denominator are integer numbers.

A permanent record may be made if a funnel which pours out a fine powder is used as a pendulum bob. If the lens of a flashlight is screened off by means of an opaque material with a small hole in it and the motion of the light dot which is formed, recorded by means of a photographic camera of which the lens is kept open in a dark room, a permanent record of the figures may be made. (The camera is placed on a horizontal surface with it lens pointing upwards at the position where the light dot is when the Y-pendulum is at rest.) Lissajous figures may also be produced by means of a computer. Those in Figure 4.5-5 were made using the *Graftool* computer program. They were printed on a laser printer. (*See the remarks at the end of Chapter 4.*)

4.5.4 Fourier analysis

Up to now only periodic phenomena which were simple harmonic (i.e. which could be described by a single sine or cosine function) were considered. In nature many oscillations occur which do not meet this requirement. Mathematically this means that such an oscillation cannot be described by means of sine and/or cosine terms in which only one angular frequency appears. Figure 4.5-7 shows an oscillation of this nature. From this graph the period (duration of one cycle)

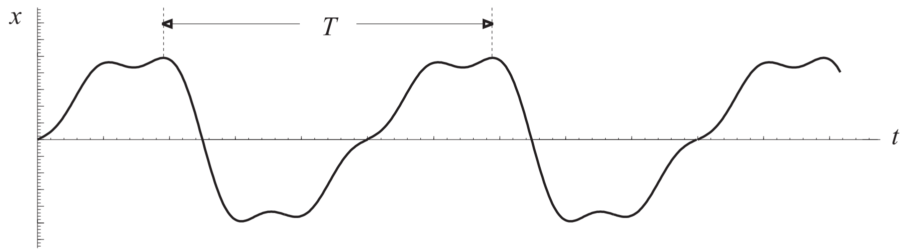


Figure 4.5-7

can be seen quite clearly, but the shape of the graph is irregular when compared to a sine curve.

The French mathematician Jean Baptiste Fourier (1769 - 1830) showed that such a periodic function may be written as the sum of a series which consists of sine and cosine terms as follows:

$$\begin{aligned} x = x(t) = a_0 \quad &+ \quad a_1 \sin \omega t + a_2 \sin 2\omega t + a_3 \sin 3\omega t + \ldots \\ &+ \quad b_1 \cos \omega t + b_2 \cos 2\omega t + b_3 \cos 3\omega t + \ldots \end{aligned} \qquad 4.5(7)$$

The process by which the **Fourier coefficients**, $a_0, a_1, a_2 \ldots, b_1, b_2, \ldots$, etc. are calculated, is known as **Fourier analysis**. This involves a procedure which is described in mathematics textbooks which deals with this subject. Excellent computer programs which can perform this task are available for personal and main frame computers.

The angular frequency ω is known as the **fundamental angular frequency** and the frequency $\nu = \omega/2\pi$ as the **fundamental frequency**. The other angular frequencies which are integer multiples of the fundamental angular frequency ω, are known as **harmonic angular frequencies** or **angular frequencies of the overtones**. Similarly the frequencies which are integer multiples of ν are called the **harmonic frequencies** or **overtone-frequencies**. The period, T, which is indicated in Figure 4.5-7 is related to the fundamental frequency by the relationship $T = 2\pi/\omega$.

The Fourier analysis of an oscillation of which the graph $x = x(t)$ has the shape of saw-teeth, is given by the sum of the infinite series

$$x(t) = 2A \sum_{i=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin n\omega t$$

Electronic signal generators which produce saw-tooth shaped potential functions are commercially available. Such potential oscillations are useful to test whether an amplifier is linear, i.e. whether it amplifies all frequencies to the same extent.

If that is the case, a saw-tooth potential function will give an output which is also in the shape of a saw-tooth. This may be tested visually by means of an oscilloscope. Deviations of the output from a saw-tooth shape, is an indication that the amplifier is not linear.

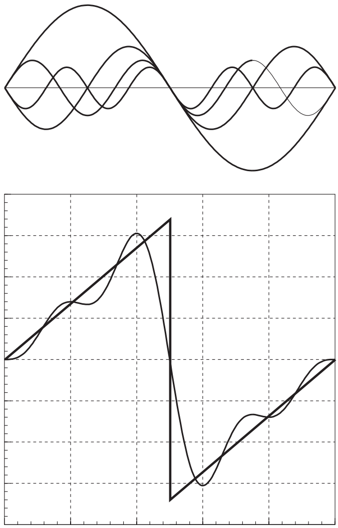


Figure 4.5-8

Figure 4.5-8 shows the first four terms of this infinite series and also their sum. Even with only four terms, indications of the saw-tooth shape can be recognised. The addition of more terms will improve the shape. Because of the factor $1/n$ which is present in all the terms, the contribution of harmonics for which n has a large value, is relatively insignificant.

By means of a Fourier analysis it is possible to analyse the sound of musical instruments and use the information to synthesise the sound by the superposition of electronic oscillations which correspond to the different Fourier components. Some electronic organs and other musical synthesisers make use of this principle.

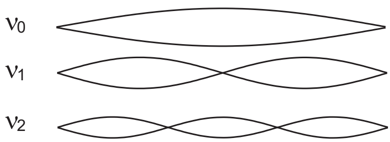


Figure 4.5-9

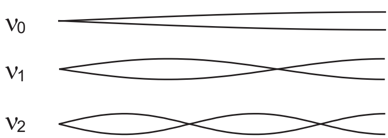


Figure 4.5-10

If notes of the same pitch are produced on different musical instruments, they do not sound the same. This fact is caused by the difference in the **quality of the sound** (also known as **timbre** which is a French word that musicians often use to describe this phenomenon). The pitch of a sound is determined by the frequency of its fundamental note and its quality by the number and relative intensities of the harmonics which are present. This is determined by the different ways in which the source of the oscillations can vibrate. The different ways in which a source can oscillate are known as its **normal modes** or **natural modes**.

That each source may have a large number of normal modes is illustrated for an oscillating string in Figure 4.5-9. For a string the ratio between the fundamental frequency and that of the harmonics is

$$\nu_0 : \nu_1 : \nu_2 \ldots = 1 : 2 : 3 \ldots$$

This relationship will be treated in the chapter which deals with waves.

As with every other system, the possible harmonics are determined by the **boundary conditions**. The boundary conditions of an oscillating string are that the displacement of the extremities must be zero at all times.

An elastic rod which is clamped at the one end only has the single boundary condition that the displacement at that position must be zero at all times. The ratio between the fundamental and harmonic frequencies is given by

$$\nu_0 : \nu_1 : \nu_2 \ldots = 1 : 3 : 5 \ldots$$

The normal modes of such a bar are shown in Figure 4.5-10.

Although it is rather difficult to imagine, the fundamental and harmonics occur simultaneously when a system vibrates. In general the amplitudes of the harmonics are much smaller than that of the fundamental and they usually damp more quickly than the fundamental. All musical instruments which produce a note of the same pitch at a very low intensity, sound more or less the same because the harmonics cannot be perceived.

4.6 Electric oscillations

4.6.1 Introduction

If a net amount of electric charge crosses a given surface within a conductor, an electric current exists through the surface. The electric current is defined as

$$i = dq/dt \qquad 4.6(1)$$

in which q is the electric charge in coulomb and t the time in seconds.

If the electric current is known as a function of time, the amount of charge which flows through a given surface may be calculated as follows:

$$q = \int_0^t i\,dt \qquad 4.6(2)$$

This is the electric charge which flows in an interval of t seconds as measured from $t = 0$.

If all the conducting charges (the charges constituting the current) have the same sign and they move in the same sense (e.g. from left to right) across the surface, the current is called a **direct current**. If each charge oscillates about an equilibrium position, the current is called an **alternating current**.

Electric circuits generally contain three kinds of passive elements which are known as **capacitance** (C), **resistance** (R) and **inductance** (L) respectively. In a resistor electromagnetic energy is irreversibly converted to heat. A capacitor stores energy in an electric field and an inductor stores energy in a magnetic field.

The electric potential difference across the terminals of a capacitor depends on the amount of electric charge, q, on it and is given by

$$v_C = q/C \qquad 4.6(3)$$

The SI units of capacitance are **farads (F)**. Farads are the same as coulombs per volt.

The electric potential difference across a resistor is determined by the current which exists in it and is given by

$$v_R = iR \qquad 4.6(4)$$

The SI units of resistance are **ohms** (Ω) which may be defined as volts per ampère.

The potential difference across the terminals of an inductor depends on the rate at which the current in it changes and is given by

$$v_L = L(di/dt) \qquad 4.6(5)$$

The SI units of inductance are **henries** (H), which are the same as volts-second per ampère.

Some of these definitions are rather unsatisfactory and will be treated in greater detail during the study of electricity and magnetism. They are, however, sufficient for the study of oscillations in electrical circuits.

4.6.2 The L-C circuit

The circuit consists of a capacitor with capacitance of C farads which is initially charged and connected in series with an inductor of L henry. The circuit contains no resistance, an arrangement which can only be attained by the use of superconductors. What we are about to derive will be a good approximation

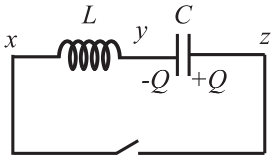


Figure 4.6-1

even if the circuit contains resistance in which the heat conversion is negligible during the time interval under consideration.

Consider a case in which the initial electric charge on the capacitor is given by $q(0) = Q$ with terminal z charged positively. Terminal y is thus at a lower electric potential as z. When the switch is closed, the current increases in the inductor from x to y. Since no potential source exists in the circuit, the potential difference between x and z will be equal to zero.

If i is the current in the inductor, we may write that

$$L\frac{di}{dt} - \frac{q}{C} = 0 \qquad \text{or} \qquad \frac{di}{dt} = \frac{1}{LC}q \qquad 4.6(6)$$

The current in the inductor is given by $i = -dq/dt$ in which q is the charge on the capacitor. The negative sign is necessary since the current in the inductor is caused by the *decrease* of the charge on the capacitor. If this relationship is used in Equation 4.6(6), it becomes

$$d^2q/dt^2 = -(1/LC)q \qquad 4.6(7)$$

This differential equation is that of a harmonic oscillator. With the given boundary conditions the following could be a solution:

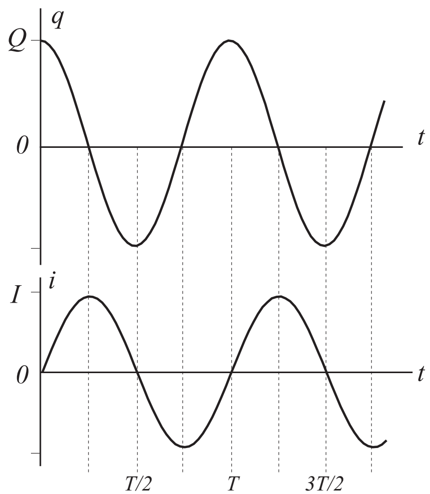


Figure 4.6-2

$$q = Q\cos\omega t, \text{ with} \qquad 4.6(8)$$

$$\omega = 2\pi\nu = 2\pi/T = (LC)^{-1/2} \qquad 4.6(9)$$

As is the case with mechanical harmonic oscillators, the frequency is independent of both the amplitude (Q in this case) and the phase constant (ϕ). As shown in Equation 4.6(9), it depends only on the capacitance and the inductance.

Using the relationship $i = -dq/dt$, the current in the inductor may be calculated from Equation 4.6(8).

$$i = \omega Q\sin\omega t = I\sin\omega t \qquad 4.6(10)$$

in which $I = \omega Q$ is the maximum value of the current.

The charge on the capacitor and the current in the inductor are illustrated graphically as functions of time in Figure 4.6-2. The phase difference between the functions is $\pi/2$ radians which corresponds to a time interval of $T/4 = (\pi/2)(LC)^{1/2}$. The charge function **leads** the current function by $\pi/2$. It may also be said that current **lags** the charge by $\pi/2$.

Initially the charge on the capacitor is a maximum and the current in the inductor equal to zero. As the charge on the capacitor decreases, the current increases and after time $T/4$, the charge is equal to zero and the current a maximum. After a further time interval $T/4$, the *magnitude* of the charge on the capacitor again reaches a maximum, but the polarity is the opposite of the initial charge. The current in the inductor is then equal to zero. After a complete cycle as measured from $t = 0$, the conditions are the same as the initial conditions.

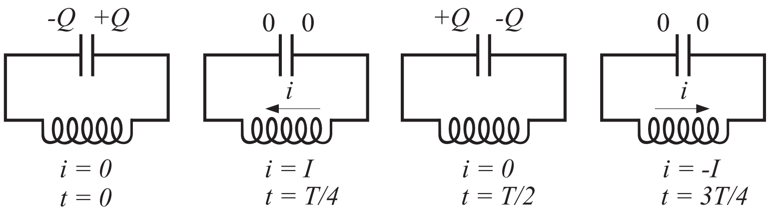


Figure 4.6-3

The functions for the current in the inductor and the charge on the capacitor are shown in Figure 4.6-2. At instants $t = T/2, 2T/2, 3T/2$, etc. the current in the inductor and therefore its magnetic field is equal to zero. All the energy is stored in the electric field of the capacitor. At instants $t = T/4, 3T/4, 5T/4$, the charge on the capacitor is zero and no electric field exists. At these instants the magnitude of the current and the magnetic field in the inductor has a maximum value and all the energy is stored in this field.

In the theory of electricity and magnetism it will be shown that the energy stored in the electric field of a capacitor is given by $E = \frac{1}{2}q^2/C$. The energy stored in the magnetic field of the inductor is equal to $E = \frac{1}{2}Li^2$ in which the symbols have the usual meaning. From the principle of energy conservation it follows that the total energy of the system should be equal to the initial energy stored in the capacitor.

$$\frac{1}{2}q^2/C + \frac{1}{2}Li^2 = \frac{1}{2}Q^2/C$$

from which the instantaneous current in the inductor may be calculated in terms of the charge on the capacitor at the same instant.

$$i = \pm[(1/LC)(Q^2 - q^2)]^{1/2} \qquad 4.6(11)$$

The charge on the capacitor and the current in the inductor are shown at intervals of one quarter of a period in Figure 4.6-3.

It is interesting to note that a perfect analogue exists between a mass of m kilogram which oscillates under the influence of the force $F = -kx$ and the electromagnetic oscillator which was discussed above. The corresponding quantities are given in the table below.

Mass of m kilogram with force $F = -kx$		Circuit containing capacitance C and inductance L	
position	x	charge	q
velocity	$v = dx/dt$	current	$i = dq/dt$
acceleration	$a = dv/dt$	rate of change in current	di/dt
mass	m	inductance	L
force constant	k	reciprocal capacitance	$1/C$
Elastic E_p	$\frac{1}{2}kx^2$	electric E_p	$\frac{1}{2}(1/C)q^2$
kinetic energy	$\frac{1}{2}mv^2$	magnetic energy	$\frac{1}{2}Li^2$
$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$		$\frac{1}{2}Li^2 + \frac{1}{2}q^2/C = \frac{1}{2}Q^2/C$	
$v = \pm[(k/m)(A^2 - x^2)]^{1/2}$		$i = \pm[(1/LC)(Q^2 - q^2)]^{1/2}$	
$d^2x/dt^2 = -(k/m)x$		$d^2q/dt^2 = -(1/LC)q$	
$x = A\cos\omega t$		$q = Q\cos\omega t$	
$\omega = 2\pi\nu = 2\pi/T = (k/m)^{1/2}$		$\omega = 2\pi\nu = 2\pi/T = (1/LC)^{1/2}$	
$T = 2\pi(m/k)^{1/2}$		$T = 2\pi(LC)^{1/2}$	
$\nu = (1/2\pi)(k/m)^{1/2}$		$\nu = (1/2\pi)(1/LC)^{1/2}$	

4.6.3 The $L - R - C$ series circuit

In 4.6.1 we considered a hypothetical circuit which contains capacitance and inductance but no resistance. The absence of resistance is the reason that no electromagnetic energy is removed form the system by heat losses. The amplitude of the oscillation remains constant since no damping is present. We will now consider a system which also contains resistance.

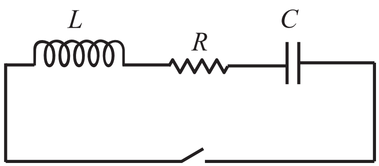


Figure 4.6-4

Figure 4.6-4 shows a circuit which contains a capacitor of C farad which is initially charged and connected in series with an inductor of L henry and a resistor of R ohm. At instant $t = 0$ the switch is closed and the charge on the capacitor is given by $q(0) = Q$.

At the instant on which the switch is closed, the sum of the potential differences across the different components must be equal to zero.

$$L(di/dt) + Ri - q/C = 0$$

If the current in the inductor is represented by i and the charge on the capacitor by q, then $i = -dq/dt$. If this relationship is used and the differential equation above is differentiated to time, it may be rewritten as follows:

$$(d^2i/dt^2) + (R/L)(di/dt) + (1/LC)i = 0 \qquad 4.6(12)$$

This equation corresponds exactly with Equation 4.4(1) for a damped oscillator. Mathematically they are equivalent in all respects and give solutions which are similar.

If the resistance in the circuit is small enough $(R/2L < (1/LC)^{1/2})$, the damping is light and the system will oscillate. The solution is as follows:

$$i = Ie^{-(R/2L))t} \sin(\omega t + \phi) \qquad 4.6(13)$$

$$\text{in which } \omega = [(1/LC) - (R^2/4L^2)]^{1/2} \qquad 4.6(14)$$

and the constants I and ϕ are determined by the initial conditions. If R is small the angular frequency, ω, will not differ much from the frequency of the corresponding $L - C$ circuit. The amplitude has exponential damping with a halving interval (half-life period) given by

$$t_{1/2} = (2L \ln 2)/R$$

A graphical representation of the solution of Equation 3.6(13) will be similar to that in Figure 4.4-1 if $\phi = 0$.

The conditions for critical damping and overdamping correspond exactly with the conditions for a mechanical oscillator which was treated in section 4.4. The reader is advised to write down these conditions for a damped $L - R - C$ circuit.

4.7 PROBLEMS CHAPTER 4

1. Consider the differential equation $d^2x/dt^2 = -9x$. Show that each of the following functions is a solution: (a) $x = 12\cos 3t$, (b) $x = -20\sin(3t - \pi)$, (c) $x = R\cos(3t + 5)$, (d) $x = 3\sin(3t - 5) - 8\cos(3t + 7)$, (e) $x = 6\sin 3t + 3\cos 3t$, (f) $x = 8\exp(3it)$, (g) $x = 15\exp(-i[3t + 5])$. In the last two functions $i = (-1)^{1/2}$ and $\exp n$ is a different notation for e^n in which e is the base of natural logarithms.

2. Consider solutions (a), (b), (c), (f) and (g) of the differential equation in question 1. Calculate in each case the angular frequency, the frequency, period, amplitude and the phase constant of the solution.

3. Show that the solution $x = A\sin(\omega t + \phi)$ of the differential equation for SHM may be written in the form $x = B\sin\omega t + C\cos\omega t$ and calculate the values of B and C in terms of the amplitude and the phase constant. Consider solution (e) in question 1. Rewrite it in the form $x = A\sin(\omega t + \phi)$. What is the amplitude of the solution in question 1(e)?

4. Consider the oscillatory function $y = 3\sin(3\pi t - \pi/4)$. Calculate the phase at the following values of the time: $t = 0$; 0,25; 0,50; 0,75; 1,00; 1,25 seconds.

5. Use the relationship $a = v(dv/dx)$ and Newton's second law to eliminate F from the force function $F = -kx$. Show that the integration of the differential equation, which is the result of this elimination, leads to the energy equation of the oscillator. In the determination of the limits of the integrals, the values of x and v which correspond to each other will have to be considered.

6. In equation 4.2(9) the function $v = v(x)$ is given for a mass of m kilograms under the action of the force law $F = -kx$. Use the relationship $v = dx/dt$ to eliminate v from this function. Show that the differential equation which is the result, is separable. Show that the integration of the equation leads to the function $x = x(t)$ which describes a harmonic oscillator.

7. If it is required by a consumer, the municipality of Pretoria will supply him with a so-called **three-phase** electrical mains. This means that the system consists of three live mains conductors which carry alternating voltages that are given by the following functions respectively: $v_1 = 353,6\sin(100\pi t)$, $v_2 = 353,6\sin(100\pi t + 2\pi/3)$ and $v_3 = 353,6\sin(100\pi t + 4\pi/3)$ volts. Draw the three phasors needed to generate numerical values of these functions on one and the same diagram as they would appear at instant $t = 0$ and then calculate the numerical value of each at this instant. Where possible, indicate the values on the diagram. If the consumer connects an appliance between the conductors with potential v_1 and v_3, the potential difference is $v_4 = v_1 - v_3$. Draw the

phasor which will generate numerical values of the function $-v_3$ and hence the one which will generate v_4. The potential difference v_4 may be described by the function $v_4 = V \sin(\omega t + \phi)$. Use the phasor diagram or a calculation to determine the numerical values of V, ϕ and ω.

8. The motion of a mass of 2 kg is subject to the force $\bar{F} = -3x\hat{x}$ in which F is measured in newtons and x in metres. Set up the differential equation for the motion and solve it for an amplitude of 4 metres and the following initial conditions: (a) $x(0) = 4$, (b) x(0) = -4, (c) $x(0) = 0$ and $\dot{x}(0) > 0$, (d) $x(0) = 0$ and $\dot{x}(0) < 0$, (e) $x(0) = -2$ and $\dot{x}(0) > 0$, (f) $x(2) = 4$, (g) $x(1) = 2$ and $\dot{x}(0) < 0$.

9. The motion of a harmonic oscillator is described by the function $x = 3\sin(\pi t + \pi/3)$ in which x is measured in metres and t in seconds. (a) What are the units of π in the term πt? What are the units of the term $\pi/3$? (b) What are the initial conditions for the motion? (c) Calculate the angular frequency, the frequency, the period, the amplitude and the initial phase of the motion. (d) Calculate the velocity and the acceleration of the oscillator as functions of time. (e) Calculate the position, velocity and acceleration at instant $t = 2/3$ s.

10. A particle executes SHM at a frequency of 50 Hz. The amplitude is 0,005 m. Write its position as a function of time if the initial position is zero and the initial velocity positive. Write the velocity and acceleration as functions of time and then as functions of position. Calculate the velocity and acceleration at the two extremities and also the middle of the straight line along which it moves.

11. A mass is suspended from the lower free end of a helical spring and executes SHM at a frequency of 2 Hz. The amplitude of the motion is 0,3 m. Calculate (a) the time which elapses while it moves from the equilibrium position directly to a position 0,15 m below it, (b) the maximum values of the speed and the magnitude of the acceleration.

12. A particle executes SHM at a frequency of 80 Hz and an amplitude of 0,002 m. Calculate the phase when the position is (a) 0,001 m, (b) -0,001 m. Calculate in each case the smallest possible positive value if it is assumed that the phase constant is zero.

13. A particle with mass 0,20 kg is suspended from the lower free end of a helical spring. If it is displaced to a position 0,15 m below the equilibrium position and set free to oscillate, the frequency is 0,5 Hz. (a) Calculate the speed of the particle when it passes through the equilibrium position. (b) Calculate the magnitude of the acceleration when it is 0,05 m above the equilibrium position. (c) Calculate the time which elapses while the particle moves downwards from a position 0,075 m above the equilibrium position, directly to one which is 0,050 m below it. (d) By how much will the spring contract if the mass is removed?

14. A mass of 0,20 kg is suspended from the free lower end of a helical spring of which the force constant is 30 $\mathrm{N\,m^{-1}}$. While it hangs in equilibrium, the mass is struck from below with a sturdy rod so that it moves vertically upwards. When the mass is 0,02 m above the equilibrium position, it leaves the rod at a speed of 0,5 $\mathrm{m\,s^{-1}}$. Use the principle of energy conservation to calculate the amplitude of the ensuing simple harmonic motion if friction may be disregarded. Calculate the frequency of the oscillation.

15. One end of a steel knitting needle is clamped in a vice and the other end displaced and allowed to oscillate. Assume that the free end executes SHM. The amplitude of the motion is 0,002 m. When the oscillating free end passes through the equilibrium position, it has a speed of 2 $\mathrm{m\,s^{-1}}$. Calculate the frequency of the motion. Write the position as a function of time with the initial position equal to zero and the initial velocity positive. Write the velocity as a function of time and then as a function of the position.

16. A particle executes linear SHM with an amplitude of 0,02 m. At the extremities of the straight line along which it moves, the magnitude of its acceleration is 2 $\mathrm{m\,s^{-2}}$. Calculate the angular frequency and the frequency of the motion. Write the position as a function of time with the initial position equal to zero and the initial velocity negative.

17. When a mass of 2 kg is suspended from the free end of a helical spring, it extends 0,08 m. The mass of 2 kg is removed and replaced with one of 0,5 kg. At what frequency will it oscillate after a vertical displacement from the equilibrium position?

18. A mass of 2 kg is suspended from the free end of a helical spring of which the force constant is 32 $\mathrm{N\,m^{-1}}$. It is displaced vertically and allowed to oscillate with an amplitude of 0,01 m along a straight line. Calculate the amount of energy which the system possesses as a result of the oscillation. Using the equilibrium position as origin, draw a graph of the kinetic energy as a function of position. Draw a graph of the elastic potential energy of the spring as a function of position on the same diagram. Indicate the total energy on the graph.

19. When a person with a mass of 75 kg gets into his car, it is lowered by 10 mm. The mass of the car is 1500 kg. Calculate the natural frequency with which the car and the person will oscillate after an initial vertical displacement if damping may be disregarded.

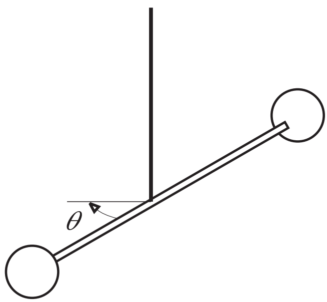
20. Calculate the length of a mathematical pendulum which has a period of (a) 1 s, (b) 2 s at a place where $g = 9,8\ \mathrm{m\,s^{-2}}$.

21. A thin uniform homogeneous straight rod has a length of 2 m and a mass

of 6 kg. It is pivoted at one end so that it can swing in a vertical plane. Friction may be disregarded. $g = 10\ \mathrm{m\,s^{-2}}$. Use the specified quantities in the calculations which follow. (a) Calculate the moment of inertia of the rod about the pivot. (b) Set up the differential equation which describes its motion and show that it may be approximated by SHM if the amplitude is relatively small. (c) Give solutions for the equation for an amplitude of 0,1 radians and the initial condition $\theta(0) = 0{,}1$ rad. (d) Calculate the period and the frequency of the motion. (e) Calculate the angular velocity and the angular acceleration of the rod as functions of time and also as functions of the angular displacement, θ.

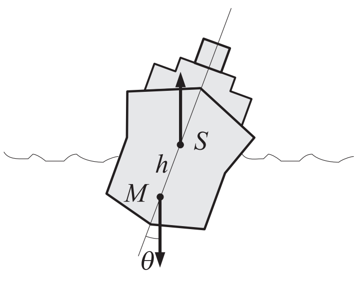
22. A hoop of which the material thickness may be disregarded in the calculation of its moment of inertia, has a mass of 2 kg and a radius of 1 m. $g = 10\ \mathrm{m\,s^{-2}}$. Calculate its period and equivalent mathematical pendulum length if it oscillates in the vertical plane containing the hoop if (a) it is suspended at its perimeter on a knife-edge, (b) suspended by means of a light cord of which the length is 1 m.

23.



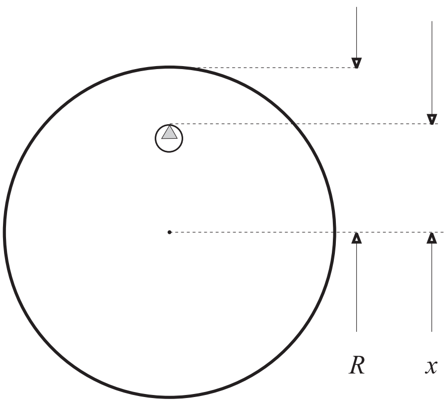
Two bodies which may be approximated by point masses, have a mass of 1 kg each and are fixed to the extremities of a light rod of length 0,8 m. The centre of the rod is fixed to a steel wire of which the torsion constant is 8 $\mathrm{N\,m\,rad^{-1}}$. Calculate the moment of inertia of the system about the wire from which it is suspended. Write the restoring torque as a function of the angular displacement θ. Show that the system will execute SHM for all values of the angular displacement. Write a solution for the differential equation for an amplitude of 60°. Calculate the period of the motion.

24.



A ship displaces V m^3 of sea-water of which the density is $\rho\ \mathrm{kg\,m^{-3}}$. The distance between the centre of mass, M, and the **metacentre**, S, (the point on the ship's vertical plane of symmetry through which the buoyant force acts) is h metres. Calculate the torque which acts on the ship if the roll angle (the angular deviation from vertical) is θ. Set up the differential equation for the roll motion of the ship and show that it is approximately SHM if the amplitude is relatively small. Disregard damping due to the viscosity of the water and air. The moment of inertia of the vessel about its roll axis is I kg m^2. Give a possible solution for the differential equation and calculate the period of the motion in terms of the given quantities. Discuss the advantages and disadvantages of (a) a large metacentric distance, (b) a small metacentric distance.

25.



A solid cylindrical disk has a radius of R metres. A small hole is drilled through it at distance x from the centre parallel to its axis. The disk is pivoted in this hole so that it can oscillate in the vertical plane which contains it. (a) Calculate the period of the oscillation. (b) Calculate the value of x for which the period will be a minimum. (c) Draw a graph of the period as a function of x which describes pivot positions along a given diameter. In the calculation of the points needed for the graph a numerical value may be assigned to R.

26. When an object moves through a **fluid** (liquid or gas) it experiences a frictional force due to the viscosity of the fluid. The direction of this force which is known as **drag**, is always opposite to the direction in which the body moves. If the body is spherical and the flow of the fluid around it is laminar, the drag is given by **Stokes's Law**: $F = 6\pi\eta r v$ in which η is the coefficient of friction of the fluid, r the radius of the sphere and v its speed through the fluid. An inflated rubber balloon with its contents has a mass of 0,01 kg and a radius of 0,25 m. It is suspended from the lower free end of a light helical spring of which the force constant is 10 N m^{-1}. It is displaced vertically from the equilibrium position and allowed to oscillate. The temperature is 60° at which the coefficient of viscosity of air is 2×10^{-5} N s m^{-2} (Pa s). Set up the differential equation for the motion of the balloon and give a suitable solution. Calculate the period of the oscillation and also the halving time (half-life period) of the damping. Disregard frictional effects on the spring and the buoyant force on the balloon. Make a calculation to establish whether it is a realistic assumption that the buoyant force on the balloon may be disregarded.

27. The fundamental frequency of a taut violin string is given by $\nu_0 = (T/\mu)^{1/2}/2L$ in which T is the tension in the string in newtons, μ its mass per unit length in kilograms per metre and L its length in metres. Two identical strings of equal length are both tuned to give a fundamental frequency of 256 Hz (middle-C on the scientific scale). The tension of one of the strings is increased in such

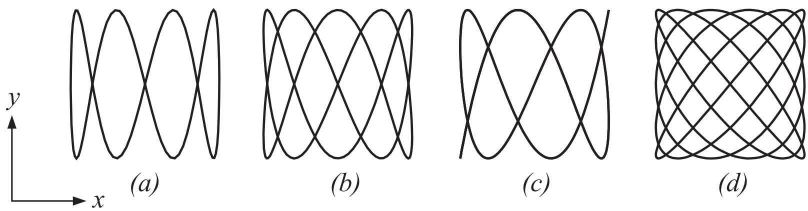
a way that beats with a frequency of 10 Hz are observed when they vibrate simultaneously. Calculate the ratio of their tensions under these circumstances.

28. Two tuning forks with frequencies 440 Hz and 444 Hz respectively are allowed to vibrate simultaneously. What is the frequency of the note that an observer will hear? What is the beat frequency?

29. The pairs of oscillations given below are superimposed. Calculate in each case the amplitude of the resulting oscillation and also its phase difference with $\sin \omega t$. (a) $x_1 = 3 \sin \omega t$ and $x_2 = 4 \sin \omega t$; (b) $x_1 = 3 \cos \omega t$ and $x_2 = 4 \sin \omega t$; (c) $x_1 = 3 \sin \omega t$ and $x_2 = 4 \cos(\omega t + 3\pi/2)$; (d) $x_1 = 2 \sin \omega t$ and $x_2 = 6 \cos(\omega t + \pi/4)$

30. Two simple harmonic motions with periods 0,010 and 0,011 s, have amplitudes of 0,1 and 0,5 m respectively. The oscillations are superimposed. They are parallel and are in phase at a given instant. Calculate the time interval which has to elapse before they will be in phase again. Also calculate the maximum and minimum values of the amplitude of the resulting oscillation.

31.



Determine in each case the ratio between ω_x and ω_y for the Lissajous figures in the sketch above.

32. The Fourier expansion for an oscillation of which the graph of each cycle is a rectangular shape, is as follows:

$$x = x(t) = \sum_{n=1}^{\infty} (n - 0,5)^{-1} \sin(2n-1)t$$

Sketch at least one cycle of the function taking only the first four terms of the series into account. If the reader has access to a computer on which a suitable graph-plotting program is installed, this function and many others should be studied by plotting them. A large variety of Fourier series is given in The Handbook of Mathematical Tables and Formulas by Murray Spiegel (Shaum Outline Series - McGraw-Hill).

33. Calculate the frequency of an oscillator circuit which consists of a 10 nF capacitor and a 1 mH inductor.

34. An L-C-R series circuit is connected as shown in Figure 4.6-4. The inductance is 10 mH, the capacitance 2 μF and the resistor, 1 Ω. Calculate the natural frequency of the circuit and the halving time (half-life period) of the damping. If the capacitance and the inductance are kept the same, calculate the value of the resistor which will result in critical damping.

In the section *The Amateur Scientist* of the journal *Scientific American* (Vol 228, no 3, August 1973) pp 107 to 109 a complete description is given for the construction of apparatus by means of which Lissajous figures can be produced. The constructions are fairly simple and effective.

ANSWERS TO PROBLEMS

CHAPTER 1

1(a) $\bar{A} = 200\hat{x} + 0\hat{y}$ km; $\bar{B} = 300,9\hat{x} + 399,3\hat{y}$ km; $\bar{C} = 0\hat{x} + 300\hat{y}$ km; $\bar{D} = -779,4\hat{x} - 450\hat{y}$ km; (b) 200 km east; 640,6 km E $38,56°$N; 860,2 km E $54,39°$N; 373,8 km W $41,83°$N; (c) 373,8 km W $41,83°$N; (d) 373,8 km E $41,83°$S; (e) 1900 km; (f) 200 kmh^{-1}; (g) 39,35 kmh^{-1} W 41,84°N; (h) 200 kmh^{-1} E 53°N.
2(a) 11 m; (b) 3 m north; (c) 0,55 ms^{-1} ;(d) 0,15 ms^{-1} north.
3(a) 0,6 ms^{-1} north; (b) $\bar{0}$ ms^{-1}; (c) 2 ms^{-1} south; (d) $\bar{0}$ ms^{-2}; (e) 0,6250 ms^{-1};(f) 0,1250 ms^{-1} north; (g) B and C; D and E.
4. N,N; N,E; N,S; E,S; S,N.
5(a) 4,0 + 8,0 + 1,6 + 3,6 + 12,0 + 9,0 = 38,2 m; (b) 1 ms^{-1} south; (c) 0 ms^{-2}; (d) 2 ms^{-2} north; (e) 5 ms^{-2} south.
6. +1; +3;-1;-3 ms^{-1} and +2, +4, +5, +3 m.
7(a) $-25\hat{x}$ m; (b) $-15\hat{x}$ m; (c) t = 1 s; t = 5 s; (d) $\bar{v} = (30 - 10t)\hat{x}$ m s^{-1}; (e) $30\hat{x}$ ms^{-1}, t = 3 s;(g) $\bar{a} = -10\hat{x}$ m s^{-2}.
8(a) $3\hat{x} + 2\hat{y} + \hat{z}$ metres; (b) 3,742 metres; (c) $-2\hat{x} - 2\hat{y} - \hat{z}$ m s^{-1}; (d) 3 m s^{-1}; (e) $6\hat{x} + 3\hat{y} - 2\hat{z}$ ms^{-2}; (f) 7 m s^{-2}; (g) $4\hat{x} + 1,5\hat{y} - \hat{z}$ m.
9(a) $(2t^2)\hat{x} + (t^2 - 4t)\hat{y} + (3t - 5)\hat{z}$ m; (b) $-5\hat{z}$ m; (c) $(4t)\hat{x} + (2t - 4)\hat{y} + 3\hat{z}$ ms^{-1} (d) $-4\hat{y} + 3\hat{z}$ ms^{-1} ; (e) 5 ms^{-1}; (f) $4\hat{x} + 2\hat{y}$ ms^{-2}; (g) $18(14)^{-1/2} = 4,811$ ms^{-1}.
10(a) A right circular cone with radius 2 m and a pitch of 1,5 metre per revolution. (b) $(8\pi\cos 4\pi t)\hat{x} - (8\pi\sin 4\pi t)\hat{y} + 3\hat{z}$ ms^{-1}; (c) 25,31 ms^{-1}; (d) 315,8 ms^{-2}
11(a) $-10\hat{x}$ ms^{-1}; (b) $t = -2,5$ s; (c) $-4\hat{x}$ ms^{-2}; (d) $(-2t^2 - 10t - 12)\hat{x}$ metre; (e) at origin; (f) $-2\hat{x}$ ms^{-1}.
12(a) 6 ms^{-1}; (b) $(3\cos 2t)\hat{x} + (3\sin 2t)\hat{y}$ m; (c) $(-12\cos 2t)\hat{x} + (-12\sin 2t)\hat{y}$ ms^{-2}; (d) $x^2 + y^2 = 9$; (e) $\bar{v}\bullet\bar{r} = 0$
13(a) $-2\hat{x} + 6\hat{y} + 3\hat{z}$ ms^{-2}; (b) $(-t^2 - t + 1)\hat{x} + (3t^2 - 2t + 3)\hat{y} + (1,5t^2 - 2t - 2)\hat{z}$ metre; (c) t = 2 s; t = -2/3 s; (d) $\arccos(-16/21) = 139,63°$.
14(a) $(2t - 3)\hat{z}$ ms^{-1}; (b) t = 1,5 s; (c) t = 4 s; (d) t = - 1 s; (e) t = -2 and t = 5 s; (f) invalid question; (g) $(t^2 - 3t - 10)\hat{z}$ m; (h) t = 5 and t = -2 s; (i) $0\hat{z}$; $4\hat{z}$; $6\hat{z}$ m; (j) $v = \pm(4z + 49)^{1/2}\hat{z}$ ms^{-1}.
15(a) 9 ms^{-2}; (b) $(4t - 3)\hat{x} + (7t + 6)\hat{y} + (-4t + 2)\hat{z}$ ms^{-1}; (c) 7 ms^{-1}; (d) 69,56°; (e) $(2t^2 - 3t + 2)\hat{x} + (3,5t^2 + 6t - 2)\hat{y} + (-2t^2 + 2t + 1)\hat{z}$ m; (f) $2\hat{x} + 26\hat{y} - 4\hat{z}$

m; (g) $\arccos(-16/21) = 139,63°$.
16(a) $(-10t+50)\hat{z}$ ms^{-1}; (b) $(-5t^2+50t)\hat{z}$ m; (c) t= 5 s; 125 m; (d) t= 2 and t = 8 s; (e) t = 3 s; (f) t = 7 s; (g) $-50\hat{z}$ ms^{-1}; (h) 12 s; (i) $\bar{v} = \pm(-20z+2500)^{1/2}\hat{z}$ m s^{-1}
17(a) $y = -5t^2 + 80t$ m; (b) $t = 8$ s; (c) 320 m; (d) $(-10t + 80)\hat{y}$ m s^{-1}; (e) $80\hat{y}$ ms^{-1}; (f) $\bar{0}$ ms^{-1}.
18. 40,2 ms^{-1}. 19. 5,4 ms^{-1}
20(a) $(v_o \cos\theta)\hat{x}+(v_o \sin\theta)\hat{y}$; (b) $(v_o \cos\theta)\hat{x}+(v_o \sin\theta-gt)\hat{y}$ m s^{-1}; (c) $(v_o \cos\theta)t\hat{x}+(-0,5gt^2+[v_o \sin\theta]t)\hat{y}$ m; (d) $y = (\tan\theta)\hat{x}-(g\sec^2\theta/2v_o^2)x^2$; (e) $(v-0^2 \sin 2\theta)/g$ m; 45°; (f) $(v_o^2 \sin^2\theta)/2g$ m; (g) $\arctan(\tan\theta - [g\sec\theta/v_o]t)$; $\arctan(\tan\theta - [g\sec^2\theta/v_o^2)]x$; $t = (v_o \sin\theta)/g$; $x = (v_o^2 \sin 2\theta)/g$.
21(a) $-10\hat{y}$ ms^{-2}, (b) $38,30\hat{x}+32,14\hat{y}$ ms^{-1}; (c) $38,30\hat{x}+(-10t+32,14)\hat{y}$ ms^{-1}; (d) $38,30t\hat{x}+(-5t^2+32,14t)\hat{y}$ m; (e) $v_y = \pm(-20y+1033)^{1/2}$; (f) 51,65 m, (g) $y = -3,409\times10^{-3}x^2+0,8392x$; (h) 246,2 m; (i) 6,428 s.
22(a) 60,00 m; (b) 6,928 m; (c) 138,6 m; If $\hat{x}$ is forwards and $\hat{y}$ upwards, then $y = 1,73x - x^2/80$.
23(a) $34,64\hat{x}+(20-10t)\hat{y}$ ms^{-1}; (b) $34,64t\hat{x}+(20t-5t^2)\hat{y}$ m; (c) $y = 0,5774x - 0,004167x^2$ m; (d) 207,8 m; (e) $-49°6'$.
24(a) $40\hat{x}+(-10t+40)\hat{y}$ ms^{-1}; (b) $(40t)\hat{x}+(-5t^2+40t)\hat{y}$ m; (c) 6 s; (d) 20 m; (e) 240 m; (f) 80 m.
25(a) 20 s; (b) 223,6 ms^{-1}; (c) 2000 m.
26. 30°; 30 s; 1125 m (low trajectory) and 60°; 51,96 s; 3375 m (high trajectory)
27(a) $37,59\hat{x}+(-10t+13,68)\hat{y}$ ms^{-1}; $37,59t\hat{x}+(-5t^2+13,68t)\hat{y}$ m; (b) $y = -3,539\times10^{-3}x^2+0,3640x$; (c) $\theta = \arctan(-7,078\times10^{-3}x+0,3640)$; (d) 115,1 m; (e) -24,27°.
28.$-13,79\hat{x}+5,898\hat{y}$ ms^{-1} 30. $d\theta/dt = 1,2\cos^2\theta$; 0,3 rad s^{-1}
29. $\bar{r} = 8t\hat{x}+(-5t^2+6t)\hat{y}$ m; $\bar{v} = 8\hat{x}+(-10t+6)\hat{y}$ ms^{-1}; $\bar{a} = 0\hat{x}+(-10)\hat{y}$ ms^{-2}
31. $-0,182\hat{y}$ ms^{-1}; $\bar{a} = -L^2\hat{y}/4(L^2-x^2)^{3/2} = -L^2\hat{y}/4(L^2-0,25t^2)^{3/2}$ ms^{-2}
32(a) $(2t)\hat{x}+(16t^2)\hat{y}$ m; (b) 31,04 ms^{-2}; (c) 7,779 ms^{-2}
34. $-1,666\hat{x}+5,688\hat{y}$ ms^{-1}; 5,927 ms^{-1}; 106,3°.
33. 125 m; 75 ms^{-1}; 30 ms^{-2} 35. $41,67\hat{y}$ m; $25\hat{y}$ ms^{-1}; $10\hat{y}$ ms^{-2}.
36. 5 ms^{-2}; $\ln(5/3) = 0,511$ s.
37(a) $22\hat{x}$ ms^{-1}; 22 ms^{-1}; (b) $18\hat{x}$ ms^{-l}; 18 m s^{-1} (c) $20\hat{x}+2\hat{y}$ m s^{-1}; 20,1 m s^{-1}
38. $27,50\hat{x}+12,99\hat{y}$ km h^{-1}; 3,42 km; 0,238 h.
39. 204 kmh^{-1}, S 11,30°W. 40. 351,4°, 130,9 km $^{-1}$h
41. E 2,018°S, 259,97 kmh^{-1}
42(a) $(3-7t)\hat{x}'+2\hat{y}'+4\hat{z}'$ m; (b) $(3-2t)\hat{x}'+(2+t)\hat{y}'+(4+2t)\hat{z}'$ m.
43(a) $\bar{r} = (6t+2)\hat{x}+(-2t-1)\hat{y}+(3t+2)\hat{z}$ m; (b) $\bar{r}' = 2\hat{x}'-\hat{y}'+2\hat{z}'$ m; (c) (i) $\bar{v}' = -\hat{x}'-2\hat{y}'+3\hat{z}'$ ms^{-1}; $\bar{r}' = (-t+2)\hat{x}'+(-2t-1)\hat{y}'+(3t+2)\hat{z}'$ m; (ii) $\bar{v}' = 4\hat{x}'-\hat{y}'+\hat{z}'$ ms^{-1}; $\bar{r}' = (4t+2)\hat{x}'+(-t-1)\hat{y}'+(t+2)\hat{z}'$. N.B $t = t'$
44(a) $\bar{r} = (3t)\hat{x}+(2t-0,9t^2)\hat{y}$ m; (b) $\bar{r}' = (2t-0,9t^2)\hat{y}$ m; (c) parabola: $y = 2x/3 - x^2/10$; straight vertical line, maximum height 1,111 m.
45. 3 kmh^{-1}

46(a) $\bar{r}'(0) = \hat{x}' + 2\hat{y}' + 3\hat{z}'$ m; $\bar{r}'(2) = -5\hat{x}' + 2\hat{y}' + 3\hat{z}'$ m; $\bar{r}'(10) = -29\hat{x}' + 2\hat{y}' + 3\hat{z}'$ m; $\bar{r}'(100) = -299\hat{x}' + 2\hat{y}' + 3\hat{z}'$ m. (b) Up to eight significant figures no difference can be found from the previous answers.
47(a) $\bar{r}'(0) = \hat{x}' + 2\hat{y}' + 3\hat{z}'$ m; $\bar{r}'(2) = (-4,8 \times 10^3)\bar{x}' + 2\hat{y}' + 3\hat{z}'$ m; $\bar{r}'(10) = (-2,4 \times 10^9)\hat{x}' + 2\hat{y}' + 3\hat{z}'$ m; $\bar{r}'(100) = (-2,4 \times 10^{10})\hat{x}' + 2\hat{y}' + 3\hat{z}'$ m; (b) $\hat{r}'(0) = 0,6\hat{x}' + 2\hat{y}' + 3\hat{z}'$ m; $\bar{r}'(2) = (-4,80 \times 10^8)\hat{x}' + 2\hat{y}' + 3\hat{z}'$ m; $\bar{r}'(10) = (-2,40 \times 10^9)\hat{x}' + 2\hat{y}' + 3\hat{z}'$m; $\bar{r}'(100) = (-2,40 \times 10^{10})\hat{x}' + 2\hat{y}' + 3\hat{z}'$ m.
48(a) $x_2 = 8 + 1,8 \times 10^8 t$ m; $x_1 = 7 + 1,8 \times 10^3 t$ m; (b) $x_2 = 6,4 + 1,8 \times 10^8 t$ m; $x_1 = 5,6 + 1,8 \times 10^8 t$ m. (c) the last set is correct.
49(a) $\bar{r}_1(0) = 1,5\hat{x} + 1,0\hat{y}$ m; $\bar{r}_2(0) = 3,5\hat{x} + 2,5\hat{y}$ m; (b) $\bar{r}_1(0) = 0,9\hat{x} + 2,5\hat{y}$ m; $\bar{r}_2(0) = 2,1\hat{x} + \hat{y}$ m; (c) $\theta_G = 36,87°$, $\theta_L = 51,34°$; (d) the second angle is correct.
50(a) $t'_1 = 2$ s; $t'_2 = 5$ s; (b) $t'_1 = 2,5$ s; $t'_2 = 6,25$ s; (c) Galilei: 3 s; Lorentz: 3,75 s; The second set is correct.
51. $\gamma = 1,25$; $3,25 \times 10^{-8}$ s.
52. $0,87c$ m s^{-1}
53. $0,8c$ m s^{-1}
54. $0,87c$ m s^{-1}
55. 60 m; $4,167 \times 10^{-7}$ s; $2,5 \times 10^{-7}$ s.
56(a) 81,3 mm and 70 mm; (b) 59,07 mm and 70 mm; (c) 19,11 mm and 70 mm.
57(a) 5×10^{-7} s; (b) $8,33 \times 10^{-7}$ s.
58. $0,97c\hat{x}$ m s^{-1}
59 . $0,27c\hat{x}$ m s^{-1}

CHAPTER 2

1. $-50\hat{r}$ kg m s^{-2}; $-12,5\hat{r}$ N;
2. 4500 N
3. $1,114\hat{z}$ m s^{-1}; $0,457\hat{z}$ kg m s^{-1};
4(a) $5\hat{z}$ N; (b) $2,5\hat{z}$m s^{-2} (c) $50\hat{z}$m s^{-1}
5(a) $3\hat{x}$ m s^{-2}; (b) $(3t - 9)\hat{x}$ m s^{-1}; (c) $(1,5t^2 - 9t + 7,5)\hat{x}$ m; (d) 1 s; 5 s; (e) $-24\hat{x}$ kg m s^{-1}; $24\hat{x}$ kg m s^{-1}
6(a) $6\hat{x} + 3\hat{y} - 2\hat{z}$ m s^{-2}; (b) 7 m s^{-2}; (c) $(6t - 2)\hat{x} + (3t - 2)\hat{y} + (-2t - 1)\hat{z}$ m s^{-1};(d) $(12t - 4)\hat{x} + (6t - 4)\hat{y} + (-4t - 2)\hat{z}$ kg m s^{-1}; (e) $(49t^2 - 32t + 9)^{1/2}$ m s^{-1}; (f) 3 m s^{-1}; (g) $(3t^2 - 2t + 3)\hat{x} + (1,5t^2 - 2t + 2)\hat{y} - (t^2 + t - 1)\hat{z}$ m; (h) 3,742 m; (i) 14 N.
7(a) 19,6 N; (b) 29,6 N; (c) 9,6 N.
8. 8000 N; 800 N;
9(a) 1 ms^{-2}; (b) 0,2 m.
10(a) $M + m = \mu$ kg;(b) mg N;(c) mg/μ ms^{-2}; (d) Mmg/μ N;(e) g ms^{-2}.
11(a) $M + m_1 + m_2 = \mu$ kg; (b) $(m_1 - m_2)g$; (c) $g(m_1 - m_2)/\mu$; (d) $gm_l(M + 2m_2)/\mu$; $gm_2(M + 2m_1)/\mu$.
12. $\rho A v^2$ newtons; ρv^2 pascal.
13(a) 9,77 N; 9,77 ms^{-2}; $1,63 \times 10^{-24}$ ms^{-2}; (b) $9,77 \times 10^3$ N; 9,77 ms^{-2}; $1,63 \times 10^{-21}$ ms^{-2}; (c) $9,77 \times 10^6$ N; 9,77 ms^{-2}; $1,63 \times 10^{-18}$ ms^{-2}.
14. $6,146 \times 10^{17}$ N; $3,554 \times 10^{22}$ N;
15. $-3,63 \times 10^6\hat{x}$ ms^{-1}.

16(a) $2kqy\hat{y}/(a^2+y^2)^{3/2}$ N C^{-1}; (b) $\pm 0,707a$ metres; $\pm 0,770kq/a^2$ N C^{-1}
(c) $2kqQy(a^2+y^2)^{-3/2}\hat{y}$ N C^{-1}
17. 1,72 nC;
18(a) 400 N; (b) 375,2 N; (c) 600,6 N
19. 0,192; 2,857 m s^{-2}; 28,57 N.
20(a) 5,196 N; (b) 31,85 m; (c) 2,895 s; (d) 5,150 s; (e) 12,37 m s^{-1}; (f) answer (a) only will differ.
21. 75 J; 25 W;
22. 37,5 J; 12,5 W.
23. 3J;
24. 31,5 J
25. 40 W
26. 2241 J
27(a) 6 J; (b) 14 J.
28(a) 225 J; (b) 675 J; (c) -900 J.
29. 600 W
30. $-(GmM/r^2)\hat{r}$ N; GmM/R J.
31. $15500 - 25x$ N; 3525 kJ; 30 kW;
32. 5 J ; 5 J.
33(a) 11,39 kJ; (b) 5,696 kW; (c) 2 kW; 35,250 J
34(a) $(x/a)^2 + (y/b)^2 = 1$; (b) $-m\omega^2\bar{r}$ N; (c) zero.
36. $5,839 \times 10^{-2}$ J.
37. 1,617 MW.
38(a) 10,20 ms^{-1};
(b) $2\hat{x} - 10\hat{y}$ ms^{-1}.
39(a) 4,5 c; (b) R4,50; (c) 2,5 c; (d) 0,45 c;
40. 540 m.
41(a) $0,5mv_0^2$; (b) $0,5mv_0^2$; (c) $0,5mv_0^2 - mgH$; (d) $(v_0^2 - 2gH)^{1/2}$;
(e) $0,5m(v_0^2 - gH)$; (f) $(v_0^2 - gH)^{1/2}$; (g) $\arccos(1 - 2gH/v_0^2)^{1/2}$); better: $\arctan[-(v_0^2/2gH - 1)^{-1/2}]$
42. $4\hat{x} - 3\hat{y} - 8\hat{z}$ kg m s^{-1}; 9,434 N s;
43. 55,56 m s^{-1}; 1389 J.
44. $29\hat{x}$ ms^{-1}; 1620 J.
45. 9,063 and 4,226 ms^{-1}; elastic; zero.
46. 0 ms^{-1}; 33 J.
47. $-5,5\hat{r}$ and $-4\hat{r}$ ms^{-1}; 33 J.
48. $1,5\hat{r}$ and $-4\hat{r}$ ms^{-1}
49. $-0,1\hat{r}$ ms^{-1}; 0,1350 J.
50. $0,8\hat{r}$ and $-0,7\hat{r}$ ms^{-1}.
51. 1,414 and 1,414 ms^{-1}; 45°.
52. $-2,309\hat{y}$ ms^{-1}; $2\hat{x} + 1,155\hat{y}$ ms^{-1}.
54(a) $(4t^3+6)\hat{x} + (9t^2 - 8t + 15)\hat{y} - (3t^2+8)\hat{z}$ ms^{-1}; (b) 477 J; 3069 J;
(c) 2592 J; (d) ΔE_k = work done on particle.
55. 960 J.
56(a) 0,5 m; (b) $5(1-4y^2)^{1/2}$ m s^{-1}.
57. 1,732 ms^{-1}.
58. 707 ms^{-1}; 4990 J.
59. $3,2 \times 10^{-2}$ m; 239,7 J.
60. 0,26 m.
61. 17,35 MeV.
62. 4,946 MeV; 8,642 MeV; 69,18°.
63. 26,73 MeV.
64. 3; 212,3 MeV.
65. $6,667 \times 10^{-7}$ N; 0,6667 Pa; and $4,714 \times 10^{-7}$ N; 0,333 Pa.
66. $3,613 \times 10^8$ V.
67. 9,488 MeV.
68. $0,9950c$

CHAPTER 3

1. $4,691 \times 10^6$ m from the centre of mass of the earth.
2. $\bar{r}_a = \bar{0}$; $\bar{r}_b = 150\hat{y}$ pm; $\bar{r}_c = 300\hat{y}$ pm; $\bar{r}_d = 300\hat{x} + 150\hat{y}$ pm; $\bar{r}_e = -300\hat{x}$

$+150\hat{y}$ pm; $\bar{r}_f = 150\hat{y} + 200\hat{z}$ pm ; $\bar{R} = 0\hat{x} + 150\hat{y} + 51{,}61\hat{z}$ pm.

3. $(2 \times 10^8 t - 8)\hat{x}$ m; $(5 \times 10^7 t - 2)\hat{x}$ m; $5 \times 10^7 \hat{x}$ m s^{-1}. 4. 0,1 m

5. $2a/\pi$ metres from the centre on the axis of symmetry.

6. $4a/3\pi$ metres from the centre on the axis of symmetry.

7 $3A/8$ metres from the centre on the axis of symmetry.

8(a) 2 rad s^{-1}; π s; $1/\pi$ Hz; (b) 2,5 rad s^{-1}; $0{,}8\pi$ s; $5/4\pi$ Hz; (c) as in (b); (d) 4π rad s^{-1}; 0,5 s; 2 Hz.

9(a) 2×10^{-3} ms$-l$; (b) $2 \times 10^{-3}\hat{x}$ and -2×10^{-3} m s^{-1}; (c) $-2 \times 10^{-3}\hat{x}$ $-2 \times 10^{-3}\hat{y}$; (d) zero; (e) $2{,}828$ $\times 10^{-3}$ ms^{-1}.

10(a) $-2{,}5\pi$ rad s^{-2}; (b) 140π rad = 70 revolutions.

11(a) $-6{,}25\pi$ rad s^{-2}; (b) 8 s. 12(a) 4 rad s^{-1}; (b) 512 N.

13(a) 2 m s^{-1}; (b) 4,472 m s^{-1}. 14. 30N; 70N.

15(a) 0,9046 m; (b) 4,254 m s^{-1}; (c) 5603; Last position.

16. 7,746 ms^{-1} = 27,89 kmh^{-1}; 17. 30,96°.

18. 17,35°.

19(a) 3,927 m s^{-1}; (b) 6,169 m s^{-2}; (c) 0,6169.

20. 3,491 rad s^{-1}; will not slip; 3,845 rad s^{-1} = 36,72 revolutions per minute.

21. $3{,}948 \times 10^5$; 22. $4{,}217 \times 10^7$ m $\approx 7\times$the radius of the earth.

23(a) 7,733 km s^{-1}; (b) $5{,}419 \times 10^3$ s $\approx$ 1,5 h; (c) 8,965 ms^{-2}

24(a) 4 kg m^2; (b) 14 kg m^2; (c) 94 kg m^2; 25. 3,5 kg m^2

26. 18m kg m^2.

27(a) $3{,}520 \times 10^6$; (b) $6{,}850 \times 10^5$; (c) $2{,}925 \times 10^6$; (d) $4{,}320 \times 10^6$ all u(pm)^2

28(a) 12 kg m^2; (b) 4 kg m^2; 29(a) 3,75; (b) 9,75; (c) 5,25 all kg m^2.

30. $r_1 = m_2 r/(m_1 + m_2)$; $r_2 = m_1 r/(m_1 + m_2)$; $I = m_1 m_2 r^2/(m_l + m_2)$; $\mu = m_1 m_2/(m_1 + m_2)$

31. 255 kg; 1,5 m from thick end. 32. 1,133 m from B.

33. 2,1 m from A.

34(a) 1 kg m^2; (b) 40 N m; (c) 200 N; (d) 80 N; (e) -16 N m; (f) -16 rad s^{-2}; (g) 200 rad; (h) 5 s; (i) 3200 J.

35(a) 20 N m; (b) 50 N; (c) 2400 J.

36(a) $\pi/4$ rad s^{-1}; (b) 750 N m; (c) 589,0 watt; (d) $2{,}945 \times 10^{-3}$ m^{3}s^{-1}.

37(a) 4 kg m^2; (b) $12\hat{x}$ N m; (c) $3\hat{x}$ rad s^{-2}; (d) $(3t - 15)\hat{x}$ rad s^{-1}; (e) $(1{,}5t^2 - 15t)$ rad; (f) t = 5 s; (g) 0 J; 72 J; (h) 72 J.

38(a) 1 kg m^2; (b) $40\hat{y}$ rad s^{-1}; (c) 20 rad; (d) $40\hat{y}$ rad s^{-2}; (e) $40\hat{y}$ Nm; (f) 200 N; (g) 1000 N.

39(a) 18 kg m^2; (b) $22{,}36\hat{y}$ rad s^{-1}; (c) $5\hat{y}$ rad s^{-2}; (d) $90\hat{y}$ N m; (e) $-\hat{z}$; $-4{,}472\hat{z}$ m s^{-1}; (f) 500 N; 450 N; (g) $-50\hat{z}$ N.

40. 10 km. 41. $2v$ m s^{-1}.

42. Sphere: $1{,}1952(gh)^{1/2}$; Cylinder: $1{,}1547(gh)^{1/2}$; Hoop: $(gh)^{1/2}$ all m s^{-1}.

43. 25,13 kW.

44(a) $N/\pi(R^2 - r^2)$; (b) $2Nxdx/(R^2 - r^2)$; (c) $2\mu Nxdx/(R^2 - r^2)$; (d) $2\mu Nxdx/(R^2 - r^2)$ (e) $\tau = 4\mu N(R^3 - r^3)/3(R^2 - r^2)$; (f) $P = \tau\,\omega$

45. $4{,}711 \times 10^6$ revolutions second^{-1}. 46. 42,21 km s^{-1}

47. $-2,179 \times 10^{-18}$J $= -13,60$ eV; $-5,488 \times 10^{-19}$J $= -3,400$ eV; 2,467 PHz; 3,4 V.

CHAPTER 4

2. The angular frequencies, frequencies and periods are the same for all: 3 rad s^{-1}, $3/2\pi$ Hz, $2\pi/3$ s. Their amplitudes and phase constants are as follows: (a) 12; 0; (b) 20; $-\pi$; (c) R; 5; (f) 8; 0; (g) 15; -5.
3. $B = A\cos\phi$; $C = A\sin\phi$; $x = 6,708\sin(3t + 0,4636)$.
4. $-\pi/4$; $\pi/2$; $5\pi/4$; 2π; $11\pi/4$; $14\pi/4$.
7. 0; 306,2 and -306,2 volts; $v_4 = 612,5\sin(100\pi t + \pi/6)$ volts.
8. All solutions are of the form $x = 4\sin(1,225t + \phi)$ with the following values for ϕ for the different solutions: (a) $(1+4n)\pi/2$; (b) $(3+4n)\pi/2$; (c) $2n\pi$; (d) $(1+2n)\pi$; (e) $(11/3+4n)\pi/2$; (f) $(1+4n)\pi/2 - 2,449$; (g) $(5/3+4n)\pi/2 - 1,225$.

9(a) rad s^{-1}; rad; (b) $x(0) = 2,598$; $\dot{x}(0) = 4,712$; (c) π rad s^{-1}; 0,5 Hz; 2 s; 3 m and $\pi/3$ rad; (d) $x(2/3) = 0$; $\dot{x}(2/3) = -3\pi$; $\ddot{x}(2/3) = 0$.

10. $x(t) = 0,005\sin(100\pi t + 2n\pi)$; $\dot{x}(t) = 0,5\pi\cos(100\pi t + 2n\pi)$; $\ddot{x}(0) = -50\pi^2\sin(100\pi t + 2n\pi)$; $\dot{x}(x) = \pm 100(2,5\times 10^{-5} - x^2)^{1/2}\pi$; $\ddot{x}(x) = -10^4\pi^2 x$; $\dot{x}(0,005) = 0$; $\ddot{x}(0,005) = -50\pi^2$; $\dot{x}(0) = \pm 0,5\pi$; $\ddot{x}(0) = 0$.

11(a) 0,041 s; (b) $1,2\pi$ m s^{-1}; $4,8\pi^2$ m s^{-2}.

12. If the motion is described by means of a sine function: (a) $\pi/6 + 2n\pi$ and also $5\pi/6 + 2n\pi$ (b) $7\pi/6 + 2n\pi$ and also $11\pi/6 + 2n\pi$. If the motion is described by means of a cosine function, subtract $\pi/2$ in each case.

13(a) 0,4712 m s^{-1}; (b) 0,4935 m s^{-2}; (c) 0,2749 s; (d) 1,013 m.

14. 0,045 m; 1,95 Hz.
15. 159 Hz; $0,002\sin(1000t)$; $2\cos(1000t)$; $\pm 1000(4\times 10^{-6} - x^2)^{1/2}$.
16. 10 rad s^{-1}; 1,592 Hz; $0,02\sin(10t + \pi)$ m.
17. 3,559 Hz.
18. $1,600 \times 10^{-3}$ J.
19. 1,098 Hz.

20(a) 0,2482 m; (b) 0,9929 m.

21(a) 8 kg m^2; (c) $0,1\cos(2,739t)$; (d) 2,29 s; 0,44 Hz; (e) $-0,274\sin(2,739t)$; $-0,75\cos(2,739t)$; $\pm 2,74(0,01 - \theta^2)^{1/2}$; $-7,5\theta$.

22(a) 2,81 s; 2 m; (b) 3,14 s; 2,5 m.

23. 0,32 kg m^2; 1,257 s.
24. $6,28(I/Vgh\rho)^{1/2}$ s.

25(a) $2\pi[(R^2 + x^2)/2gx]^{1/2}$; (b) $x = 0,7071R$

26. 0,1987 s; 147,2 s.
27. 0,9262 or 1,0797 s.
28. 442 Hz; 4 Hz.

29(a) 7; zero; (b) 5; 0,6435; (c) 7, zero; (d) 4,799; 2,057

30. 0,1100 s; 0,6 m; 0,4 m.
31. 1:4; 2:5; 3:7; 5:6.
33. 50,33 kHz.
34. 1,125 kHz; $1,386 \times 10^{-2}$ s; 141,4 Ω.